4.5 Termination

Termination problems:

Given a finite TRS $R$ and a term $t$, are all $R$-reductions starting from $t$ terminating?

Given a finite TRS $R$, are all $R$-reductions terminating?

**Proposition 4.21** Both termination problems for TRSs are undecidable in general.

**Proof.** Encode Turing machines using rewrite rules and reduce the (uniform) halting problems for TMs to the termination problems for TRSs. \(\square\)

Consequence:
Decidable criteria for termination are not complete.

**Reduction Orderings**

Goal:

Given a finite TRS $R$, show termination of $R$ by looking at finitely many rules $l \rightarrow r \in R$, rather than at infinitely many possible replacement steps $s \rightarrow_R s'$.

A binary relation $\sqsubseteq$ over $T_\Sigma(X)$ is called **compatible with $\Sigma$-operations**, if $s \sqsubseteq s'$ implies $f(t_1, \ldots, s, \ldots, t_n) \sqsubseteq f(t_1, \ldots, s', \ldots, t_n)$ for all $f \in \Omega$ and $s, s', t_i \in T_\Sigma(X)$.

**Lemma 4.22** The relation $\sqsubseteq$ is compatible with $\Sigma$-operations, if and only if $s \sqsubseteq s'$ implies $t[s]_p \sqsubseteq t[s']_p$ for all $s, s', t \in T_\Sigma(X)$ and $p \in \text{pos}(t)$.

Note: compatible with $\Sigma$-operations = compatible with contexts.

A binary relation $\sqsubseteq$ over $T_\Sigma(X)$ is called **stable under substitutions**, if $s \sqsubseteq s'$ implies $s\sigma \sqsubseteq s'\sigma$ for all $s, s' \in T_\Sigma(X)$ and substitutions $\sigma$.

A binary relation $\sqsubseteq$ is called a **rewrite relation**, if it is compatible with $\Sigma$-operations and stable under substitutions.

Example: If $R$ is a TRS, then $\rightarrow_R$ is a rewrite relation.

A strict partial ordering over $T_\Sigma(X)$ that is a rewrite relation is called rewrite ordering.

A well-founded rewrite ordering is called reduction ordering.

**Theorem 4.23** A TRS $R$ terminates if and only if there exists a reduction ordering $\succ$ such that $l \succ r$ for every rule $l \rightarrow r \in R$. 

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Proof. “if”: \( s \rightarrow_R s' \) if and only if \( s = t[l\sigma]_p, s' = t[r\sigma]_p \). If \( l \succ r \), then \( l\sigma \succ r\sigma \) and therefore \( t[l\sigma]_p \succ t[r\sigma]_p \). This implies \( \neg R \subseteq \succ \). Since \( \succ \) is a well-founded ordering, \( \neg R \) is terminating.

“only if”: Define \( \succ = \rightarrow^+_R \). If \( \rightarrow_R \) is terminating, then \( \succ \) is a reduction ordering. \( \square \)

Simplification Orderings

The proper subterm ordering \( \triangleright \) is defined by \( s \triangleright t \) if and only if \( s/p = t \) for some position \( p \neq \varepsilon \) of \( s \).

A rewrite ordering \( \succ \) over \( T_\Sigma(X) \) is called simplification ordering, if it has the subterm property: \( s \triangleright t \) implies \( s \succ t \) for all \( s, t \in T_\Sigma(X) \).

Example:

Let \( R_{\text{emb}} \) be the rewrite system \( R_{\text{emb}} = \{ f(x_1, \ldots, x_n) \rightarrow x_i \mid f \in \Omega, 1 \leq i \leq n = \text{arity}(f) \} \).

Define \( \triangleright_{\text{emb}} = \rightarrow^+_R_{\text{emb}} \) and \( \succeq_{\text{emb}} = \rightarrow^*_R_{\text{emb}} \) (“homeomorphic embedding relation”).

\( \triangleright_{\text{emb}} \) is a simplification ordering.

Lemma 4.24 If \( \succ \) is a simplification ordering, then \( s \triangleright_{\text{emb}} t \) implies \( s \succ t \) and \( s \succeq_{\text{emb}} t \) implies \( s \succeq t \).

Proof. Since \( \succ \) is transitive and \( \succeq \) is transitive and reflexive, it suffices to show that \( s \rightarrow_{R_{\text{emb}}} t \) implies \( s \succ t \). By definition, \( s \rightarrow_{R_{\text{emb}}} t \) if and only if \( s = s[l\sigma] \) and \( t = s[r\sigma] \) for some rule \( l \rightarrow r \in R_{\text{emb}} \). Obviously, \( l \triangleright r \) for all rules in \( R_{\text{emb}} \), hence \( l \succ r \). Since \( \succ \) is a rewrite relation, \( s = s[l\sigma] \succ s[r\sigma] = t \). \( \square \)

Goal:

Show that every simplification ordering is well-founded (and therefore a reduction ordering).

Note: This works only for finite signatures!

To fix this for infinite signatures, the definition of simplification orderings and the definition of embedding have to be modified.

Theorem 4.25 (“Kruskal’s Theorem”) Let \( \Sigma \) be a finite signature, let \( X \) be a finite set of variables. Then for every infinite sequence \( t_1, t_2, t_3, \ldots \) there are indices \( j > i \) such that \( t_j \succeq_{\text{emb}} t_i \). (\( \succeq_{\text{emb}} \) is called a well-partial-ordering (wpo).)

Proof. See Baader and Nipkow, page 113–115. \( \square \)
Theorem 4.26 (Dershowitz) If \( \Sigma \) is a finite signature, then every simplification ordering \( \succ \) on \( T_\Sigma(X) \) is well-founded (and therefore a reduction ordering).

Proof. Suppose that \( t_1 \succ t_2 \succ t_3 \succ \ldots \) is an infinite descending chain.

First assume that there is an \( x \in \text{var}(t_{i+1}) \setminus \text{var}(t_i) \). Let \( \sigma = [t_i/x] \), then \( t_{i+1}\sigma \geq x\sigma = t_i \) and therefore \( t_i = t_i\sigma \succ t_{i+1}\sigma \succeq t_i \), contradicting reflexivity.

Consequently, \( \text{var}(t_i) \supseteq \text{var}(t_{i+1}) \) and \( t_i \in T_\Sigma(\mathcal{V}) \) for all \( i \), where \( \mathcal{V} \) is the finite set \( \text{var}(t_1) \). By Kruskal’s Theorem, there are \( i < j \) with \( t_i \succeq_{\text{emb}} t_j \). Hence \( t_i \not\succeq t_j \), contradicting \( t_i \succ t_j \).

There are reduction orderings that are not simplification orderings and terminating TRSs that are not contained in any simplification ordering.

Example:

Let \( R = \{ f(f(x)) \rightarrow f(g(f(x))) \} \).

\( R \) terminates and \( \rightarrow_{\mathcal{R}}^{+} \) is therefore a reduction ordering.

Assume that \( \rightarrow_{\mathcal{R}} \) were contained in a simplification ordering \( \succ \). Then \( f(f(x)) \rightarrow_{\mathcal{R}} f(g(f(x))) \) implies \( f(f(x)) \succ f(g(f(x))) \), and \( f(g(f(x))) \succeq_{\text{emb}} f(f(x)) \) implies \( f(g(f(x))) \succeq f(f(x)) \), hence \( f(f(x)) \succ f(f(x)) \).

Recursive Path Orderings

Let \( \Sigma = (\Omega, \Pi) \) be a finite signature, let \( \succ \) be a strict partial ordering (“precedence”) on \( \Omega \).

The lexicographic path ordering \( \succ_{\text{lpo}} \) on \( T_\Sigma(X) \) induced by \( \succ \) is defined by: \( s \succ_{\text{lpo}} t \) iff

1. \( t \in \text{var}(s) \) and \( t \neq s \), or
2. \( s = f(s_1, \ldots, s_m), t = g(t_1, \ldots, t_n) \), and
   a. \( s_i \succeq_{\text{lpo}} t \) for some \( i \), or
   b. \( f \succ g \) and \( s \succ_{\text{lpo}} t_j \) for all \( j \), or
   c. \( f = g, s \succ_{\text{lpo}} t_j \) for all \( j \), and \( (s_1, \ldots, s_m) \succ_{\text{lpo}} (t_1, \ldots, t_n) \).

Lemma 4.27 \( s \succ_{\text{lpo}} t \) implies \( \text{var}(s) \supseteq \text{var}(t) \).

Proof. By induction on \( |s| + |t| \) and case analysis. \( \square \)

Theorem 4.28 \( \succ_{\text{lpo}} \) is a simplification ordering on \( T_\Sigma(X) \).
Proof. Show transitivity, subterm property, stability under substitutions, compatibility with $\Sigma$-operations, and irreflexivity, usually by induction on the sum of the term sizes and case analysis. Details: Baader and Nipkow, page 119/120.

Theorem 4.29 If the precedence $\succ$ is total, then the lexicographic path ordering $\succ_{lpo}$ is total on ground terms, i.e., for all $s, t \in T_\Sigma(\emptyset)$: $s \succ_{lpo} t \lor t \succ_{lpo} s \lor s = t$.

Proof. By induction on $|s| + |t|$ and case analysis.

Recapitulation:

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering ("precedence") on $\Omega$. The lexicographic path ordering $\succ_{lpo}$ on $T_\Sigma(X)$ induced by $\succ$ is defined by: $s \succ_{lpo} t$ iff

1. $t \in \text{var}(s)$ and $t \neq s$, or
2. $s = f(s_1, \ldots, s_m), t = g(t_1, \ldots, t_n)$, and
   a. $s_i \succ_{lpo} t$ for some $i$, or
   b. $f \succ g$ and $s \succ_{lpo} t_j$ for all $j$, or
   c. $f = g, s \succ_{lpo} t_j$ for all $j$, and $(s_1, \ldots, s_m) \succ_{lpo} \text{lex}(t_1, \ldots, t_n)$.

There are several possibilities to compare subterms in (2)(c):

- compare list of subterms lexicographically left-to-right ("lexicographic path ordering (lpo)", Kamin and Lévy)
- compare list of subterms lexicographically right-to-left (or according to some permutation $\pi$)
- compare multiset of subterms using the multiset extension ("multiset path ordering (mpo)", Dershowitz)

To each function symbol $f$ with $\text{arity}(n) \geq 1$ associate a status $\in \{\text{mul}\} \cup \{\text{lex}_\pi \mid \pi : \{1, \ldots, n\} \to \{1, \ldots, n\}\}$ and compare according to that status ("recursive path ordering (rpo) with status")

The Knuth-Bendix Ordering

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a strict partial ordering ("precedence") on $\Omega$, let $w : \Omega \cup X \to \mathbb{R}_0^+$ be a weight function, such that the following admissibility conditions are satisfied:

- $w(x) = w_0 \in \mathbb{R}_0^+$ for all variables $x \in X$; $w(c) \geq w_0$ for all constants $c \in \Omega$.
- If $w(f) = 0$ for some $f \in \Omega$ with $\text{arity}(f) = 1$, then $f \succeq g$ for all $g \in \Omega$. 

The weight function $w$ can be extended to terms as follows:

$$w(t) = \sum_{x \in \text{var}(t)} w(x) \cdot \#(x,t) + \sum_{f \in \Omega} w(f) \cdot \#(f,t).$$

The Knuth-Bendix ordering $\succ_{\text{kbo}}$ on $T_\Sigma(X)$ induced by $\succ$ and $w$ is defined by: $s \succ_{\text{kbo}} t$ iff

1. $\#(x,s) \ge \#(x,t)$ for all variables $x$ and $w(s) > w(t)$, or
2. $\#(x,s) \ge \#(x,t)$ for all variables $x$, $w(s) = w(t)$, and
   a) $t = x$, $s = f^n(x)$ for some $n \ge 1$, or
   b) $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$, and $f \succ g$, or
   c) $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$, and $(s_1, \ldots, s_m) \succ_{\text{kbo}} (t_1, \ldots, t_m)$.

**Theorem 4.30** The Knuth-Bendix ordering induced by $\succ$ and $w$ is a simplification ordering on $T_\Sigma(X)$.

**Proof.** Baader and Nipkow, pages 125–129. \qed