This does not hold for superposition:

Let \( b \succ c \succ d \). Assume that the current rewrite system (representing the current interpretation) contains the rule \( c \rightarrow d \). Now consider the clause \( b \approx c \lor b \approx d \).

We need a further inference rule to deal with clauses of this kind, either the “Merging Paramodulation” rule of Bachmair and Ganzinger or the following “Equality Factoring” rule due to Nieuwenhuis:

\[
\text{Equality Factoring: } \frac{C \lor s \approx t \lor s \approx t}{C' \lor t \neq t'}
\]

Note: This inference rule subsumes the usual factoring rule.

How do the non-ground versions of the inference rules for superposition look like?

Main idea as in the resolution calculus:

Replace identity by unifiability. Apply the mgu to the resulting clause. In the ordering restrictions, replace \( \succ \) by \( \not\preceq \).

However:

As in Knuth-Bendix completion, we do not want to consider overlaps at or below a variable position.

Consequence: there are inferences between ground instances \( D\theta \) and \( C\theta \) of clauses \( D \) and \( C \) which are not ground instances of inferences between \( D \) and \( C \).

Such inferences have to be treated in a special way in the completeness proof.
The Superposition Calculus – Formally

Until now, we have seen most of the ideas behind the superposition calculus and its completeness proof.

We will now start again from the beginning giving precise definitions and proofs.

Inference rules are applied with respect to the commutativity of equality $\approx$.

Inference rules:

**Pos. Superposition:**

\[
\frac{D' \lor t \approx t' \quad C' \lor s[u] \approx s'}{(D' \lor C' \lor s[t'] \approx s')\sigma}
\]

where $\sigma = \text{mgu}(t, u)$ and $u$ is not a variable.

**Neg. Superposition:**

\[
\frac{D' \lor t \approx t' \quad C' \lor s[u] \not\approx s'}{(D' \lor C' \lor s[t'] \not\approx s')\sigma}
\]

where $\sigma = \text{mgu}(t, u)$ and $u$ is not a variable.

**Equality Resolution:**

\[
\frac{C' \lor s \not\approx s'}{C'\sigma}
\]

where $\sigma = \text{mgu}(s, s')$.

**Equality Factoring:**

\[
\frac{C' \lor s' \approx t' \lor s \approx t}{(C' \lor t \not\approx t' \lor s \approx t')\sigma}
\]

where $\sigma = \text{mgu}(s, s')$.

**Theorem 4.44** All inference rules of the superposition calculus are correct, i.e., for every rule

\[
\frac{C_n, \ldots, C_1}{C_0}
\]

we have $\{C_1, \ldots, C_n\} \models C_0$.

**Proof.** Exercise. \[\Box\]
Orderings:

Let $\succ$ be a reduction ordering that is total on ground terms.

To a positive literal $s \approx t$, we assign the multiset $\{s, t\}$, to a negative literal $s \nRightarrow t$ the multiset $\{s, s, t, t\}$. The literal ordering $\succ_L$ compares these multisets using the multiset extension of $\succ$.

The clause ordering $\succ_C$ compares clauses by comparing their multisets of literals using the multiset extension of $\succ_L$.

Inferences have to be computed only if the following ordering restrictions are satisfied:

- In superposition inferences, after applying the unifier to both premises, the left premise is not greater than or equal to the right one.
- The last literal in each premise is maximal in the respective premise, i.e., there exists no greater literal (strictly maximal for positive literals in superposition inferences, i.e., there exists no greater or equal literal).
- In these literals, the lhs is not smaller than the rhs (in superposition inferences: neither smaller nor equal).

A ground clause $C$ is called redundant w.r.t. a set of ground clauses $N$, if it follows from clauses in $N$ that are smaller than $C$.

A clause is redundant w.r.t. a set of clauses $N$, if all its ground instances are redundant w.r.t. $G_\Sigma(N)$.

The set of all clauses that are redundant w.r.t. $N$ is denoted by $Red(N)$.

$N$ is called saturated up to redundancy, if the conclusion of every inference from clauses in $N \setminus Red(N)$ is contained in $N \cup Red(N)$.

Superposition: Refutational Completeness

For a set $E$ of ground equations, $T_\Sigma(\emptyset)/E$ is an $E$-interpretation (or $E$-algebra) with universe $\{[t] \mid t \in T_\Sigma(\emptyset)\}$.

One can show (similar to the proof of Birkhoff’s Theorem) that for every ground equation $s \approx t$ we have $T_\Sigma(\emptyset)/E \models s \approx t$ if and only if $s \leftrightarrow^* _E t$.

In particular, if $E$ is a convergent set of rewrite rules $R$ and $s \approx t$ is a ground equation, then $T_\Sigma(\emptyset)/R \models s \approx t$ if and only if $s \downarrow_R t$. By abuse of terminology, we say that an equation or clause is valid (or true) in $R$ if and only if it is true in $T_\Sigma(\emptyset)/R$. 
Construction of candidate interpretations (Bachmair & Ganzinger 1990):

Let \( N \) be a set of clauses not containing \( \bot \). Using induction on the clause ordering we define sets of rewrite rules \( E_C \) and \( R_C \) for all \( C \in G_\Sigma(N) \) as follows:

Assume that \( E_D \) has already been defined for all \( D \in G_\Sigma(N) \) with \( D \prec_C C \). Then \( R_C = \bigcup_{D \prec_C C} E_D \).

The set \( E_C \) contains the rewrite rule \( s \rightarrow t \), if

(a) \( C = C' \cup s \approx t \).
(b) \( s \approx t \) is strictly maximal in \( C \).
(c) \( s \succ t \).
(d) \( C \) is false in \( R_C \).
(e) \( C' \) is false in \( R_C \cup \{s \rightarrow t\} \).
(f) \( s \) is irreducible w. r. t. \( R_C \).

In this case, \( C \) is called productive. Otherwise \( E_C = \emptyset \).

Finally, \( R_\infty = \bigcup_{D \in G_\Sigma(N)} E_D \).

**Lemma 4.45** If \( E_C = \{s \rightarrow t\} \) and \( E_D = \{u \rightarrow v\} \), then \( s \succ u \) if and only if \( C \succ_C D \).

**Corollary 4.46** The rewrite systems \( R_C \) and \( R_\infty \) are convergent.

**Proof.** Obviously, \( s \succ t \) for all rules \( s \rightarrow t \) in \( R_C \) and \( R_\infty \).

Furthermore, it is easy to check that there are no critical pairs between any two rules: Assume that there are rules \( u \rightarrow v \) in \( E_D \) and \( s \rightarrow t \) in \( E_C \) such that \( u \) is a subterm of \( s \). As \( \succ \) is a reduction ordering that is total on ground terms, we get \( u \prec s \) and therefore \( D \prec_C C \) and \( E_D \subseteq R_C \). But then \( s \) would be reducible by \( R_C \), contradicting condition (f).

**Lemma 4.47** If \( D \nprec_C C \) and \( E_C = \{s \rightarrow t\} \), then \( s \succ u \) for every term \( u \) occurring in a negative literal in \( D \) and \( s \succeq v \) for every term \( v \) occurring in a positive literal in \( D \).

**Corollary 4.48** If \( D \in G_\Sigma(N) \) is true in \( R_D \), then \( D \) is true in \( R_\infty \) and \( R_C \) for all \( C \succ_C D \).

**Proof.** If a positive literal of \( D \) is true in \( R_D \), then this is obvious.

Otherwise, some negative literal \( s \not\approx t \) of \( D \) must be true in \( R_D \), hence \( s \not\downarrow_{R_D} t \). As the rules in \( R_\infty \setminus R_D \) have left-hand sides that are larger than \( s \) and \( t \), they cannot be used in a rewrite proof of \( s \downarrow t \), hence \( s \not\downarrow_{R_C} t \) and \( s \not\downarrow_{R_\infty} t \).
Corollary 4.49 If $D = D' \lor u \approx v$ is productive, then $D'$ is false and $D$ is true in $R_\infty$ and $R_C$ for all $C \succ_C D$.

Proof. Obviously, $D$ is true in $R_\infty$ and $R_C$ for all $C \succ_C D$.

Since all negative literals of $D'$ are false in $R_D$, it is clear that they are false in $R_\infty$ and $R_C$. For the positive literals $u' \approx v'$ of $D'$, condition (e) ensures that they are false in $R_D \cup \{u \rightarrow v\}$. Since $u' \preceq u$ and $v' \preceq u$ and all rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than $u$, these rules cannot be used in a rewrite proof of $u' \downarrow v'$, hence $u' \not\downarrow_{R_C} v'$ and $u' \not\downarrow_{R_\infty} v'$.

Lemma 4.50 ("Lifting Lemma") Let $C$ be a clause and let $\theta$ be a substitution such that $C\theta$ is ground. Then every equality resolution or equality factoring inference from $C\theta$ is a ground instance of an inference from $C$.

Proof. Exercise.

Lemma 4.51 ("Lifting Lemma") Let $D = D' \lor u \approx v$ and $C = C' \lor [\neg] s \approx t$ be two clauses (without common variables) and let $\theta$ be a substitution such that $D\theta$ and $C\theta$ are ground.

If there is a superposition inference between $D\theta$ and $C\theta$ where $u\theta$ and some subterm of $s\theta$ are overlapped, and $u\theta$ does not occur in $s\theta$ at or below a variable position of $s$, then the inference is a ground instance of a superposition inference from $D$ and $C$.

Proof. Exercise.

Theorem 4.52 ("Model Construction") Let $N$ be a set of clauses that is saturated up to redundancy and does not contain the empty clause. Then we have for every ground clause $C\theta \in G_\Sigma(N)$:

(i) $E_{C\theta} = \emptyset$ if and only if $C\theta$ is true in $R_{C\theta}$.

(ii) If $C\theta$ is redundant w. r. t. $G_\Sigma(N)$, then it is true in $R_{C\theta}$.

(iii) $C\theta$ is true in $R_\infty$ and in $R_D$ for every $D \in G_\Sigma(N)$ with $D \succ_C C\theta$.

Proof. We use induction on the clause ordering $\succ_C$ and assume that (i)–(iii) are already satisfied for all clauses in $G_\Sigma(N)$ that are smaller than $C\theta$. Note that the “if” part of (i) is obvious from the construction and that condition (iii) follows immediately from (i) and Corollaries 4.48 and 4.49. So it remains to show (ii) and the “only if” part of (i).