Corollary 4.49 *If $D = D' \vee u \approx v$ is productive, then D' is false and D is true in R_∞ and R_C for all $C \succ_C D$.*

Proof. Obviously, D is true in R_∞ and R_C for all $C \succ_C D$.

Since all negative literals of D' are false in R_D, it is clear that they are false in R_∞ and R_C. For the positive literals $u' \approx v'$ of D', condition (e) ensures that they are false in $R_D \cup \{u \to v\}$. Since $u' \preceq u$ and $v' \preceq u$ and all rules in $R_\infty \setminus R_D$ have left-hand sides that are larger than u, these rules cannot be used in a rewrite proof of $u' \downarrow v'$, hence $u' \not\downarrow_{R_C} v'$ and $u' \not\downarrow_{R_\infty} v'$. □

Lemma 4.50 ("Lifting Lemma") *Let C be a clause and let θ be a substitution such that $C\theta$ is ground. Then every equality resolution or equality factoring inference from $C\theta$ is a ground instance of an inference from C.*

Proof. Exercise. □

Lemma 4.51 ("Lifting Lemma") *Let $D = D' \vee u \approx v$ and $C = C' \vee [\neg]\, s \approx t$ be two clauses (without common variables) and let θ be a substitution such that $D\theta$ and $C\theta$ are ground.*

If there is a superposition inference between $D\theta$ and $C\theta$ where $u\theta$ and some subterm of $s\theta$ are overlapped, and $u\theta$ does not occur in $s\theta$ at or below a variable position of s, then the inference is a ground instance of a superposition inference from D and C.

Proof. Exercise. □

Theorem 4.52 ("Model Construction") *Let N be a set of clauses that is saturated up to redundancy and does not contain the empty clause. Then we have for every ground clause $C\theta \in G_\Sigma(N)$:*

(i) $E_{C\theta} = \emptyset$ if and only if $C\theta$ is true in $R_{C\theta}$.

(ii) If $C\theta$ is redundant w. r. t. $G_\Sigma(N)$, then it is true in $R_{C\theta}$.

(iii) $C\theta$ is true in R_∞ and in R_D for every $D \in G_\Sigma(N)$ with $D \succ_C C\theta$.

Proof. We use induction on the clause ordering $\succ_C$ and assume that (i)–(iii) are already satisfied for all clauses in $G_\Sigma(N)$ that are smaller than $C\theta$. Note that the "if" part of (i) is obvious from the construction and that condition (iii) follows immediately from (i) and Corollaries 4.48 and 4.49. So it remains to show (ii) and the "only if" part of (i).

Case 1: $C\theta$ is redundant w. r. t. $G_\Sigma(N)$.

If $C\theta$ is redundant w. r. t. $G_\Sigma(N)$, then it follows from clauses in $G_\Sigma(N)$ that are smaller than $C\theta$. By part (iii) of the induction hypothesis, these clauses are true in $R_{C\theta}$. Hence $C\theta$ is true in $R_{C\theta}$.

Case 2: $x\theta$ is reducible by $R_{C\theta}$.

Suppose there is a variable x occurring in C such that $x\theta$ is reducible by $R_{C\theta}$, say $x\theta \rightarrow_{R_{C\theta}} w$. Let the substitution θ' be defined by $x\theta' = w$ and $y\theta' = y\theta$ for every variable $y \neq x$. The clause $C\theta'$ is smaller than $C\theta$. By part (iii) of the induction hypothesis, it is true in $R_{C\theta}$. By congruence, every literal of $C\theta$ is true in $R_{C\theta}$ if and only if the corresponding literal of $C\theta'$ is true in $R_{C\theta}$; hence $C\theta$ is true in $R_{C\theta}$.

Case 3: $C\theta$ contains a maximal negative literal.

Suppose that $C\theta$ does not fall into Case 1 or 2 and that $C\theta = C'\theta \;\vee\; s\theta \not\approx s'\theta$, where $s\theta \not\approx s'\theta$ is maximal in $C\theta$. If $s\theta \approx s'\theta$ is false in $R_{C\theta}$, then $C\theta$ is clearly true in $R_{C\theta}$ and we are done. So assume that $s\theta \approx s'\theta$ is true in $R_{C\theta}$, that is, $s\theta \downarrow_{R_{C\theta}} s'\theta$. Without loss of generality, $s\theta \succeq s'\theta$.

Case 3.1: $s\theta = s'\theta$.

If $s\theta = s'\theta$, then there is an *equality resolution* inference

$$\frac{C'\theta \;\vee\; s\theta \not\approx s'\theta}{C'\theta}\,.$$

As shown in the Lifting Lemma, this is an instance of an *equality resolution* inference

$$\frac{C' \;\vee\; s \not\approx s'}{C'\sigma}$$

where $C = C' \;\vee\; s \not\approx s'$ is contained in N and $\theta = \rho \circ \sigma$. (Without loss of generality, σ is idempotent, therefore $C'\theta = C'\sigma\rho = C'\sigma\sigma\rho = C'\sigma\theta$, so $C'\theta$ is a ground instance of $C'\sigma$.) Since $C\theta$ is not redundant w. r. t. $G_\Sigma(N)$, C is not redundant w. r. t. N. As N is saturated up to redundancy, the conclusion $C'\sigma$ of the inference from C is contained in $N \cup \mathit{Red}(N)$. Therefore, $C'\theta$ is either contained in $G_\Sigma(N)$ and smaller than $C\theta$, or it follows from clauses in $G_\Sigma(N)$ that are smaller than itself (and therefore smaller than $C\theta$). By the induction hypothesis, clauses in $G_\Sigma(N)$ that are smaller than $C\theta$ are true in $R_{C\theta}$, thus $C'\theta$ and $C\theta$ are true in $R_{C\theta}$.

Case 3.2: $s\theta \succ s'\theta$.

If $s\theta \downarrow_{R_{C\theta}} s'\theta$ and $s\theta \succ s'\theta$, then $s\theta$ must be reducible by some rule in some $E_{D\theta} \subseteq R_{C\theta}$. (Without loss of generality we assume that C and D are variable disjoint; so we can use the same substitution θ.) Let $D\theta = D'\theta \ \vee\ t\theta \approx t'\theta$ with $E_{D\theta} = \{t\theta \to t'\theta\}$. Since $D\theta$ is productive, $D'\theta$ is false in $R_{C\theta}$. Besides, by part (ii) of the induction hypothesis, $D\theta$ is not redundant w. r. t. $G_\Sigma(N)$, so D is not redundant w. r. t. N. Note that $t\theta$ cannot occur in $s\theta$ at or below a variable position of s, say $x\theta = w[t\theta]$, since otherwise $C\theta$ would be subject to Case 2 above. Consequently, the *left superposition* inference

$$\frac{D'\theta \ \vee\ t\theta \approx t'\theta \qquad C'\theta \ \vee\ s\theta[t\theta] \not\approx s'\theta}{D'\theta \ \vee\ C'\theta \ \vee\ s\theta[t'\theta] \not\approx s'\theta}$$

is a ground instance of a *left superposition* inference from D and C. By saturation up to redundancy, its conclusion is either contained in $G_\Sigma(N)$ and smaller than $C\theta$, or it follows from clauses in $G_\Sigma(N)$ that are smaller than itself (and therefore smaller than $C\theta$). By the induction hypothesis, these clauses are true in $R_{C\theta}$, thus $D'\theta \ \vee\ C'\theta \ \vee\ s\theta[t'\theta] \not\approx s'\theta$ is true in $R_{C\theta}$. Since $D'\theta$ and $s\theta[t'\theta] \not\approx s'\theta$ are false in $R_{C\theta}$, both $C'\theta$ and $C\theta$ must be true.

Case 4: $C\theta$ does not contain a maximal negative literal.

Suppose that $C\theta$ does not fall into Cases 1 to 3. Then $C\theta$ can be written as $C'\theta \ \vee\ s\theta \approx s'\theta$, where $s\theta \approx s'\theta$ is a maximal literal of $C\theta$. If $E_{C\theta} = \{s\theta \to s'\theta\}$ or $C'\theta$ is true in $R_{C\theta}$ or $s\theta = s'\theta$, then there is nothing to show, so assume that $E_{C\theta} = \emptyset$ and that $C'\theta$ is false in $R_{C\theta}$. Without loss of generality, $s\theta \succ s'\theta$.

Case 4.1: $s\theta \approx s'\theta$ is maximal in $C\theta$, but not strictly maximal.

If $s\theta \approx s'\theta$ is maximal in $C\theta$, but not strictly maximal, then $C\theta$ can be written as $C''\theta \ \vee\ t\theta \approx t'\theta \ \vee\ s\theta \approx s'\theta$, where $t\theta = s\theta$ and $t'\theta = s'\theta$. In this case, there is a *equality factoring* inference

$$\frac{C''\theta \ \vee\ t\theta \approx t'\theta \ \vee\ s\theta \approx s'\theta}{C''\theta \ \vee\ t'\theta \not\approx s'\theta \ \vee\ t\theta \approx t'\theta}$$

This inference is a ground instance of an inference from C. By induction hypothesis, its conclusion is true in $R_{C\theta}$. Trivially, $t'\theta = s'\theta$ implies $t'\theta \downarrow_{R_{C\theta}} s'\theta$, so $t'\theta \not\approx s'\theta$ must be false and $C\theta$ must be true in $R_{C\theta}$.

Case 4.2: $s\theta \approx s'\theta$ is strictly maximal in $C\theta$ and $s\theta$ is reducible.

Suppose that $s\theta \approx s'\theta$ is strictly maximal in $C\theta$ and $s\theta$ is reducible by some rule in $E_{D\theta} \subseteq R_{C\theta}$. Let $D\theta = D'\theta \ \vee\ t\theta \approx t'\theta$ and $E_{D\theta} = \{t\theta \to t'\theta\}$. Since $D\theta$ is productive, $D\theta$ is not redundant and $D'\theta$ is false in $R_{C\theta}$. We can now proceed in essentially the same way as in Case 3.2: If $t\theta$ occurred in $s\theta$ at or below a variable position of s, say $x\theta = w[t\theta]$, then $C\theta$ would be subject to Case 2 above. Otherwise, the *right superposition* inference

$$\frac{D'\theta \ \vee \ t\theta \approx t'\theta \qquad C'\theta \ \vee \ s\theta[t\theta] \approx s'\theta}{D'\theta \ \vee \ C'\theta \ \vee \ s\theta[t'\theta] \approx s'\theta}$$

is a ground instance of a *right superposition* inference from D and C. By saturation up to redundancy, its conclusion is true in $R_{C\theta}$. Since $D'\theta$ and $C'\theta$ are false in $R_{C\theta}$, $s\theta[t'\theta] \approx s'\theta$ must be true in $R_{C\theta}$. On the other hand, $t\theta \approx t'\theta$ is true in $R_{C\theta}$, so by congruence, $s\theta[t\theta] \approx s'\theta$ and $C\theta$ are true in $R_{C\theta}$.

Case 4.3: $s\theta \approx s'\theta$ is strictly maximal in $C\theta$ and $s\theta$ is irreducible.

Suppose that $s\theta \approx s'\theta$ is strictly maximal in $C\theta$ and $s\theta$ is irreducible by $R_{C\theta}$. Then there are three possibilities: $C\theta$ can be true in $R_{C\theta}$, or $C'\theta$ can be true in $R_{C\theta} \cup \{s\theta \to s'\theta\}$, or $E_{C\theta} = \{s\theta \to s'\theta\}$. In the first and the third case, there is nothing to show. Let us therefore assume that $C\theta$ is false in $R_{C\theta}$ and $C'\theta$ is true in $R_{C\theta} \cup \{s\theta \to s'\theta\}$. Then $C'\theta = C''\theta \ \vee \ t\theta \approx t'\theta$, where the literal $t\theta \approx t'\theta$ is true in $R_{C\theta} \cup \{s\theta \to s'\theta\}$ and false in $R_{C\theta}$. In other words, $t\theta \downarrow_{R_{C\theta} \cup \{s\theta \to s'\theta\}} t'\theta$, but not $t\theta \downarrow_{R_{C\theta}} t'\theta$. Consequently, there is a rewrite proof of $t\theta \to^* u \leftarrow^* t'\theta$ by $R_{C\theta} \cup \{s\theta \to s'\theta\}$ in which the rule $s\theta \to s'\theta$ is used at least once. Without loss of generality we assume that $t\theta \succeq t'\theta$. Since $s\theta \approx s'\theta \succ_{\mathrm{L}} t\theta \approx t'\theta$ and $s\theta \succ s'\theta$ we can conclude that $s\theta \succeq t\theta \succ t'\theta$. But then there is only one possibility how the rule $s\theta \to s'\theta$ can be used in the rewrite proof: We must have $s\theta = t\theta$ and the rewrite proof must have the form $t\theta \to s'\theta \to^* u \leftarrow^* t'\theta$, where the first step uses $s\theta \to s'\theta$ and all other steps use rules from $R_{C\theta}$. Consequently, $s'\theta \approx t'\theta$ is true in $R_{C\theta}$. Now observe that there is an *equality factoring* inference

$$\frac{C''\theta \ \vee \ t\theta \approx t'\theta \ \vee \ s\theta \approx s'\theta}{C''\theta \ \vee \ t'\theta \not\approx s'\theta \ \vee \ t\theta \approx t'\theta}$$

whose conclusion is true in $R_{C\theta}$ by saturation. Since the literal $t'\theta \not\approx s'\theta$ must be false in $R_{C\theta}$, the rest of the clause must be true in $R_{C\theta}$, and therefore $C\theta$ must be true in $R_{C\theta}$, contradicting our assumption. This concludes the proof of the theorem. □

A Σ-interpretation $\mathcal{A}$ is called *term-generated,* if for every $b \in U_{\mathcal{A}}$ there is a ground term $t \in \mathrm{T}_\Sigma(\emptyset)$ such that $b = \mathcal{A}(\beta)(t)$.

Lemma 4.53 *Let N be a set of (universally quantified) Σ-clauses and let $\mathcal{A}$ be a term-generated Σ-interpretation. Then $\mathcal{A}$ is a model of $G_\Sigma(N)$ if and only if it is a model of N.*

Proof. (⇒): Let $\mathcal{A} \models G_\Sigma(N)$; let $(\forall \vec{x} C) \in N$. Then $\mathcal{A} \models \forall \vec{x} C$ iff $\mathcal{A}(\gamma[x_i \mapsto a_i])(C) = 1$ for all γ and a_i. Choose ground terms t_i such that $\mathcal{A}(\gamma)(t_i) = a_i$; define θ such that $x_i\theta = t_i$, then $\mathcal{A}(\gamma[x_i \mapsto a_i])(C) = \mathcal{A}(\gamma \circ \theta)(C) = \mathcal{A}(\gamma)(C\theta) = 1$ since $C\theta \in G_\Sigma(N)$.

(⇐): Let $\mathcal{A}$ be a model of N; let $C \in N$ and $C\theta \in G_\Sigma(N)$. Then $\mathcal{A}(\gamma)(C\theta) = \mathcal{A}(\gamma \circ \theta)(C) = 1$ since $\mathcal{A} \models N$. □

Theorem 4.54 (Refutational Completeness: Static View) *Let N be a set of clauses that is saturated up to redundancy. Then N has a model if and only if N does not contain the empty clause.*

Proof. If $\bot \in N$, then obviously N does not have a model. If $\bot \notin N$, then the interpretation R_∞ (that is, $\mathrm{T}_\Sigma(\emptyset)/R_\infty$) is a model of all ground instances in $G_\Sigma(N)$ according to part (iii) of the model construction theorem. As $\mathrm{T}_\Sigma(\emptyset)/R_\infty$ is term generated, it is a model of N. □

So far, we have considered only inference rules that add new clauses to the current set of clauses (corresponding to the *Deduce* rule of Knuth-Bendix Completion).

In other words, we have derivations of the form $N_0 \vdash N_1 \vdash N_2 \vdash \ldots$, where each N_{i+1} is obtained from N_i by adding the consequence of some inference from clauses in N_i.

Under which circumstances are we allowed to delete (or simplify) a clause during the derivation?

A *run* of the superposition calculus is a sequence $N_0 \vdash N_1 \vdash N_2 \vdash \ldots$, such that
(i) $N_i \models N_{i+1}$, and
(ii) all clauses in $N_i \setminus N_{i+1}$ are redundant w. r. t. N_{i+1}.

In other words, during a run we may add a new clause if it follows from the old ones, and we may delete a clause, if it is redundant w. r. t. the remaining ones.

For a run, $N_\infty = \bigcup_{i \geq 0} N_i$ and $N_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} N_j$. The set N_* of all *persistent* clauses is called the *limit* of the run.

Lemma 4.55 *If $N \subseteq N'$, then $Red(N) \subseteq Red(N')$.*

Proof. Obvious. □

Lemma 4.56 *If $N' \subseteq Red(N)$, then $Red(N) \subseteq Red(N \setminus N')$.*

Proof. Follows from the compactness of first-order logic and the well-foundedness of the multiset extension of the clause ordering. □

Lemma 4.57 *Let $N_0 \vdash N_1 \vdash N_2 \vdash \dots$ be a run. Then $Red(N_i) \subseteq Red(N_\infty)$ and $Red(N_i) \subseteq Red(N_*)$ for every i.*

Proof. Exercise. □

Corollary 4.58 *$N_i \subseteq N_* \cup Red(N_*)$ for every i.*

Proof. If $C \in N_i \setminus N_*$, then there is a $k \geq i$ such that $C \in N_k \setminus N_{k+1}$, so C must be redundant w. r. t. N_{k+1}. Consequently, C is redundant w. r. t. N_*. □

A run is called *fair,* if the conclusion of every inference from clauses in $N_* \setminus Red(N_*)$ is contained in some $N_i \cup Red(N_i)$.

Lemma 4.59 *If a run is fair, then its limit is saturated up to redundancy.*

Proof. If the run is fair, then the conclusion of every inference from non-redundant clauses in N_* is contained in some $N_i \cup Red(N_i)$, and therefore contained in $N_* \cup Red(N_*)$. Hence N_* is saturated up to redundancy. □

Theorem 4.60 (Refutational Completeness: Dynamic View) *Let $N_0 \vdash N_1 \vdash N_2 \vdash \dots$ be a fair run, let N_* be its limit. Then N_0 has a model if and only if $\bot \notin N_*$.*

Proof. ($\Leftarrow$): By fairness, N_* is saturated up to redundancy. If $\bot \notin N_*$, then it has a term-generated model. Since every clause in N_0 is contained in N_* or redundant w. r. t. N_*, this model is also a model of $G_\Sigma(N_0)$ and therefore a model of N_0.

($\Rightarrow$): Obvious, since $N_0 \models N_*$. □

Superposition: Extensions

Extensions and improvements:

simplification techniques,

selection functions (as for ordered resolution),

redundancy for inferences,

basic strategies,

constraint reasoning.