Tutorials for “Automated Reasoning”
Solution to the exercise sheet 1

Exercise 1.1: (3 P)
Determine which of the following formulas are valid/satisfiable/unsatisfiable (don’t use truth tables):

(1) \((P \land Q) \rightarrow (P \lor Q)\).

Solution.

\[\begin{align*}
(P \land Q) & \rightarrow (P \lor Q) \\
\vdash & \neg(P \land Q) \lor (P \lor Q) \\
\vdash & \neg P \lor \neg Q \lor P \lor Q \\
\vdash & (\neg P \lor P) \lor (\neg Q \lor Q) \\
\vdash & \top \lor \top \\
\vdash & \top.
\end{align*}\]

For any \(\Pi\)-valuation \(A\), under which \(Q\) and \(P\) have the same value, the formula evaluates to 1, and for other valuations the formula evaluates to 0, hence the given formula is valid.

(2) \((P \lor Q) \rightarrow (P \land Q)\).

Solution.

\[\begin{align*}
(P \lor Q) & \rightarrow (P \land Q) \\
\vdash & \neg(P \lor Q) \lor (P \land Q) \\
\vdash & (\neg(P \lor Q) \lor P) \land (\neg(P \lor Q) \lor Q) \\
\vdash & ((\neg P \land \neg Q) \lor P) \land ((\neg P \land \neg Q) \lor Q) \\
\vdash & ((\neg P \lor P) \land (\neg Q \lor P)) \land ((\neg P \lor Q) \land (\neg Q \lor Q)) \\
\vdash & (\top \land (\neg Q \lor P)) \land ((\neg P \lor Q) \land \top) \\
\vdash & (\neg Q \lor P) \land (\neg P \lor Q) \\
\vdash & (Q \rightarrow P) \land (P \rightarrow Q) \\
\vdash & (Q \leftrightarrow P).
\end{align*}\]

For any \(\Pi\)-valuation \(A\), under which \(Q\) and \(P\) have the same value, the formula evaluates to 1, and for other valuations the formula evaluates to 0, hence the given formula is satisfiable, but not valid.
(3) $(\neg P \to Q) \to ((\neg P \to \neg Q) \to P)$

Solution.

$(\neg P \to Q) \to ((\neg P \to \neg Q) \to P)$

\[ \begin{align*}
&\iff \neg(\neg P \lor Q) \lor (\neg(\neg P \lor \neg Q) \lor P) \\
&\iff (P \land \neg Q) \lor ((P \land Q) \lor P) \\
&\iff (P \land \neg Q) \lor (P \land Q) \lor P \\
&\iff (-P \land -Q) \lor P \land (-P \land Q) \lor P \\
&\iff ((-P \lor P) \land (-Q \lor P)) \lor ((-P \lor P) \land (Q \lor P)) \\
&\iff (\top \land (\neg Q \lor P)) \lor (\top \land (Q \lor P)) \\
&\iff -Q \lor P \lor Q \lor P \\
&\iff \top \lor P \\
&\iff \top.
\end{align*} \]

For any $\Pi$-valuation $A$, we have $A((-P \to Q) \to ((\neg P \to \neg Q) \to P)) = A(\top) = 1$, hence the given formula is valid.

(4) $\neg(P \to \neg P)$

Solution.

\[\begin{align*}
\neg(P \to \neg P) \iff & \neg(P \lor \neg P) \\
\iff & \neg(\neg P) \\
\iff & P.
\end{align*}\]

The obtained formula is the both CNF and DNF of the original formula. Since every conjunct/disjunct of it does not contain complementary literals, the original formula is neither valid nor unsatisfiable, therefore it is satisfiable.

(5) $\neg(P \lor \neg(P \land Q))$

Solution.

\[\begin{align*}
\neg(P \lor \neg(P \land Q)) \iff & \neg P \land \neg \neg(P \land Q) \\
\iff & \neg P \land (P \land Q) \\
\iff & (\neg P \land P) \land Q \\
\iff & \top \land Q \\
\iff & \bot.
\end{align*}\]

For any $\Pi$-valuation $A$, we have $A(\neg(P \lor \neg(P \land Q))) = A(\bot) = 0$, hence the given formula is unsatisfiable.

(6) $(P \lor \neg Q) \land \neg(\neg P \to \neg Q)$

Solution.

\[\begin{align*}
(P \lor \neg Q) \land \neg(\neg P \to \neg Q) \iff & (P \lor \neg Q) \land \neg(\neg P \lor \neg Q) \\
\iff & (P \lor \neg Q) \land \neg(P \lor \neg Q) \\
\iff & ((R) \land \neg(R)) \land (R \leftrightarrow (P \lor \neg Q)) \quad \text{(R is a new prop. var.)} \\
\iff & (R \land \neg R) \land (R \leftrightarrow (P \lor \neg Q)) \\
\iff & \top \land (R \leftrightarrow (P \lor \neg Q)) \\
\iff & \bot.
\end{align*}\]

For any $\Pi$-valuation $A$, we have $A(((R) \land \neg(R)) \land (R \leftrightarrow (P \lor \neg Q))) = A(\bot) = 0$. Since we have used only satisfiablity-preserving transformations, the original formula is unsatisfiable.
Exercise 1.2: (4 P)
Let $F, G$ be propositional formulas and $P$ be a propositional variable which does not occur in $F$ nor in $G$. Prove or refute the following propositions:

1. If $F \land G$ is valid/satisfiable, then $P \land G \land (P \rightarrow F)$ is valid/satisfiable.

   Solution.

   Assume $F \land G$ is satisfiable, meaning that there exists a Π-valuation $\mathcal{A}$, s.t. $\mathcal{A} \models F \land G$. Note, that $\mathcal{A} \models F \iff \mathcal{A} \models F \land G$. Let $\mathcal{A}'$ be a Π-valuation, s.t. $\mathcal{A}'(P) = 1$ and $\mathcal{A}'$ agrees with $\mathcal{A}$ on any other propositional variable, then, since $P$ does not occur in $F$ or $G$, we have that $\mathcal{A}' \models F$ and $\mathcal{A}' \models G$, therefore $\mathcal{A}'(P \land G \land (P \rightarrow F)) = \mathcal{A}'(P) \land \mathcal{A}'(G) \land \mathcal{A}'(P \rightarrow F) = 1 \land 1 \land (1 \rightarrow 1) = 1$. So, we’ve found a Π-valuation $\mathcal{A}'$ that models the formula $P \land G \land (P \rightarrow F)$.

   Let $\mathcal{A}''$ be a Π-valuation, s.t. $\mathcal{A}''(P) = 0$, then $\mathcal{A}''(P \land G \land (P \rightarrow F)) = \mathcal{A}''(P) \land \mathcal{A}''(G \land (P \rightarrow F)) = 0 \land \mathcal{A}''(G \land (P \rightarrow F)) = 0$. So, we’ve found a Π-valuation $\mathcal{A}''$ that does not model the formula $P \land G \land (P \rightarrow F)$.

   Having the Π-valuations $\mathcal{A}'$ and $\mathcal{A}''$, we can conclude that if $F \land G$ is valid or satisfiable, then $P \land G \land (P \rightarrow F)$ is not valid but satisfiable.

2. Let $G$ be unsatisfiable and $F \models G$. Then $F \lor G$ satisfiable.

   Solution.

   $F \models G$ if $\mathcal{A} \models F \rightarrow G$, for an arbitrary Π-valuation $\mathcal{A}$. Also, $G$ is unsat., iff $\mathcal{A} \models \neg G$, for an arbitrary Π-valuation $\mathcal{A}$. These two facts give us that for an arbitrary $\mathcal{A}$:

   $$\mathcal{A} \models \neg G \text{ and } \mathcal{A} \models F \rightarrow G \iff \mathcal{A}(\neg G) = 1 \text{ and } \mathcal{A}(F \rightarrow G) = 1$$

   $$\iff \mathcal{A}(\neg G \land (F \rightarrow G)) = 1$$

   $$\iff \mathcal{A}(\neg G \land \neg F \lor G) = 1$$

   $$\iff \mathcal{A}(\neg F \land \neg G) = 1$$

   $$\iff \mathcal{A}(\neg (F \lor G)) = 1.$$

   As the Π-valuation $\mathcal{A}$ was taken arbitrary, we obtain that $\neg (F \lor G)$ is valid and, thus, $(F \lor G)$ is unsatisfiable, or, equivalently, $(F \lor G)$ is not satisfiable.

3. If $F \rightarrow G$ is valid, and $G \rightarrow H$ is satisfiable, then $F \rightarrow H$ is satisfiable.

   Solution.

   We prove the statement by contradiction.

   Assume that $F \rightarrow G$ is valid, $G \rightarrow H$ is satisfiable, but $F \rightarrow H$ is not satisfiable.

   Let $\mathcal{A}$ be an arbitrary Π-valuation.

   Since $F \rightarrow H$ is not satisfiable (or, equivalently, it is unsatisfiable), we have that $\mathcal{A}(\neg (F \rightarrow H)) = 1$, iff $\mathcal{A}(F \land \neg H) = 1$, iff $\mathcal{A} \models F$ and $\mathcal{A} \models \neg H$. As the $\mathcal{A}$ is taken arbitrary, we conclude that the formulas $F$ and $\neg H$ are valid: $\models F$ and $\models \neg H$.

   Since $F \rightarrow G$ is valid, $\mathcal{A}(F \rightarrow G) = 1$, iff $\mathcal{A}(\neg F \lor G) = 1$, iff $\mathcal{A}(\neg F) = 1$ or $\mathcal{A}(G) = 1$, iff $\mathcal{A} \models \neg F$ or $\mathcal{A} \models G$, but from what we have already shown, we know that $\mathcal{A} \models F$, hence $\mathcal{A} \models G$. As the $\mathcal{A}$ is taken arbitrary, we conclude that the formula $G$ is valid: $\models G$. 

Since $G \rightarrow H$ is satisfiable, there exists a $\Pi$-valuation $A'$ s.t. $A'(G \rightarrow H) = 1$, iff $A'(-G) = 1$ or $A'(H) = 1$, iff $A' \models -G$ or $A' \models H$, but we have already shown that $G$ is valid, hence $A' \models H$, but this contradicts the fact that $\neg H$ is valid.

Thus our assumption was wrong and the statement of the exercise holds.

4. If $F$ is satisfiable and $G$ is satisfiable, then $F \land G$ is satisfiable.

Solution.

We refute the statement by counterexample.

Let $F = Q$ and $G = \neg Q$, where $Q$ is a propositional variable. $F$ and $G$ are clearly satisfiable, but $F \land G$ is not, because $F \land G = Q \land \neg Q \models \bot$.

Exercise 1.3: (2 P)

Transform the following formula to both CNF and DNF following the conversion steps from the lecture: $((P \rightarrow Q) \lor R) \land (\neg Q \rightarrow P)$.

Solution.

1. CNF.

$$((P \rightarrow Q) \lor R) \land (\neg Q \rightarrow P) \Rightarrow^* \land (\neg P \lor Q \lor R) \land (\neg Q \lor P) \Rightarrow \land (\neg P \lor Q \lor R) \land (Q \lor P).$$

2. DNF.

$$((P \rightarrow Q) \lor R) \land (\neg Q \rightarrow P) \Rightarrow^* \land (\neg P \lor Q \lor R) \land (\neg Q \lor P) \Rightarrow \land (\neg P \lor Q \lor R) \land (Q \lor P) \Rightarrow \land (\neg P \lor Q) \lor (\neg P \lor R) \lor (Q \lor P) \lor (R \lor Q) \lor (R \lor P).$$

(We use the notation $\Rightarrow^*_K$ to denote a multiple application of $\Rightarrow_K$.)

Exercise 1.4: (1 P)

Let $F$ be a propositional formula. Show how to check its validity using an implementation of the DPLL procedure.

Solution.

A propositional formula $F$ is valid, iff $\neg F$ is unsatisfiable. The DPLL procedure is aimed to check whether a given clause set is satisfiable or not, or, equivalently, the DPLL procedure can be used as an unsatisfiability checker. Based on these observations, one can check validity of a given formula $F$ in the following way:
1. Compute $F' = \neg F$.

2. Compute $F'' = \text{CNF}(F')$, i.e. compute the CNF of $F'$.

3. If $\text{DPLL}(\emptyset, F'')$ is false, the formula $F$ is valid, otherwise $F$ – not valid.

**Challenge Problem:** (2 Bonus Points)

Let $F$ be a propositional formula which contains no occurrence of $\rightarrow$ or $\leftrightarrow$, then $F^o$ is the propositional formula obtained by replacing all occurrences of propositional variables by their negations.

The dual of $F$, which we denote here by $F^*$, is the propositional formula obtained by replacing every occurrence of $\top$ by $\bot$, every occurrence of $\bot$ by $\top$, every occurrence of $\lor$ by $\land$ and every occurrence of $\land$ by $\lor$.

Prove or refute that $F^* \models \neg F^o$.

**Solution.**

We claim that $F^* \models \neg F^o$ holds.

**Proof.** We prove the statement by the Principle of Structural Induction.

**Basic Step.** Suppose $F$ is atomic. Consider possible cases:

- $F = P$, where $P$ is a propositional variable. Then

$$
F^* = P^* = P \quad \text{(def. of $^*$)}
$$

$$
\models \neg P 
= \neg (P^o) \quad \text{(def. of $^o$)}
= \neg F^o.
$$

- $F = \top$. Then

$$
F^* = \top^* = \bot \quad \text{(def. of $^*$)}
$$

$$
\models \neg \top 
= \neg (\top^o) \quad \text{(def. of $^o$)}
= \neg F^o.
$$

- $F = \bot$. This case is similar to the previous one.

Thus, for every atomic formula $F$, we have that $F^* \models \neg F^o$.

**Induction Step.** Let $H$ and $G$ be arbitrary propositional formulas. Suppose that $H^* \models \neg H^o$ and $G^* \models \neg G^o$ (induction hypothesis), and $F = H \circ G$, where $\circ \in \{\lor, \land\}$. Consider the following cases:
• $F = H \lor G$. Then

$$F^* = (H \lor G)^* = H^* \land G^* \quad \text{(def. of $*$)}$$

$$\models \neg H^* \land \neg G^* \quad \text{(ind.hypothesis)}$$

$$\models \neg (H^* \land G^*)$$

$$= \neg (H \lor G)^* \quad \text{(def. of $^*$)}$$

$$= \neg F^*.$$ 

• $F = H \land G$. This case is similar to the previous one.

• $F = \neg H$. Then

$$F^* = (\neg H)^*$$

$$= \neg (H^*) \quad \text{(def. of $*$)}$$

$$\models \neg (\neg H^*) \quad \text{(ind.hypothesis)}$$

$$= \neg (\neg H)^* \quad \text{(def. of $^*$)}$$

$$= \neg F^*.$$ 

Now it follows by the Principle of Structural Induction that, for every propositional formula $F$, the property $F^* \models \neg F^*$ holds.