4.7 Unfailing Completion

Classical completion:

Try to transform a set E of equations into an equivalent convergent TRS.

Fail, if an equation can neither be oriented nor deleted.

Unfailing completion (Bachmair, Dershowitz and Plaisted):

If an equation cannot be oriented, we can still use *orientable instances* for rewriting.

Note: If $\succ$ is total on ground terms, then every *ground instance* of an equation is trivial or can be oriented.

Goal: Derive a *ground convergent* set of equations.

Let E be a set of equations, let $\succ$ be a reduction ordering.

We define the relation $\rightarrow_{E^\succ}$ by

$s \rightarrow_{E^\succ} t$ iff there exist $(u \approx v) \in E$ or $(v \approx u) \in E$, $p \in \mathrm{pos}(s)$, and $\sigma : X \rightarrow \mathrm{T}_\Sigma(X)$, such that $s/p = u\sigma$ and $t = s[v\sigma]_p$ *and* $u\sigma \succ v\sigma$.

Note: $\rightarrow_{E^\succ}$ is terminating by construction.

From now on let $\succ$ be a reduction ordering that is total on ground terms.

E is called ground convergent w. r. t. $\succ$, if for all ground terms s and t with $s \leftrightarrow^*_E t$ there exists a ground term v such that $s \rightarrow^*_{E^\succ} v \leftarrow^*_{E^\succ} t$. (Analogously for $E \cup R$.)

As for standard completion, we establish ground convergence by computing critical pairs.

However, the ordering $\succ$ is not total on non-ground terms. Since $s\theta \succ t\theta$ implies $s \not\preceq t$, we approximate $\succ$ on ground terms by $\not\preceq$ on arbitrary terms.

Let $u_i \doteq v_i$ $(i = 1, 2)$ be equations in E whose variables have been renamed such that $\mathrm{var}(u_1 \doteq v_1) \cap \mathrm{var}(u_2 \doteq v_2) = \emptyset$. Let $p \in \mathrm{pos}(u_1)$ be a position such that u_1/p is not a variable, σ is an mgu of u_1/p and u_2, and $u_i\sigma \not\preceq v_i\sigma$ $(i = 1, 2)$. Then $\langle v_1\sigma, (u_1\sigma)[v_2\sigma]_p\rangle$ is called a *semi-critical pair* of E with respect to $\succ$.

The set of all semi-critical pairs of E is denoted by $\mathrm{SP}_\succ(E)$.

Semi-critical pairs of $E \cup R$ are defined analogously. If $\rightarrow_R \subseteq \succ$, then $\mathrm{CP}(R)$ and $\mathrm{SP}_\succ(R)$ agree.

Note: In contrast to critical pairs, it may be necessary to consider overlaps of a rule with itself at the top. For instance, if $E = \{f(x) \approx g(y)\}$, then $\langle g(y), g(y')\rangle$ is a non-trivial semi-critical pair.

The *Deduce* rule takes now the following form:

Deduce:

$$\frac{E, \quad R}{E \cup \{s \approx t\}, \quad R} \qquad \text{if } \langle s, t\rangle \in \mathrm{SP}_{\succ}(E \cup R).$$

Moreover, the fairness criterion for runs is replaced by

$$\mathrm{SP}_{\succ}(E_* \cup R_*) \subseteq E_\infty$$

(i. e., if every semi-critical pair between persisting rules or equations is computed at some step of the derivation).

Analogously to Thm. 4.37 we obtain now the following theorem:

Theorem 4.38 *Let $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$ be a fair run; let $R_0 = \emptyset$. Then*

(1) $E_* \cup R_*$ is equivalent to E_0, and

(2) $E_* \cup R_*$ is ground convergent.

Moreover one can show that, whenever there exists a *reduced* convergent R such that $\approx_{E_0} = \downarrow_R$ and $\rightarrow_R \in \succ$, then for every fair *and simplifying* run $E_* = \emptyset$ and $R_* = R$ up to variable renaming.

Here R is called reduced, if for every $l \rightarrow r \in R$, both l and r are irreducible w. r. t. $R \setminus \{l \rightarrow r\}$. A run is called simplifying, if R_* is reduced, and for all equations $u \approx v \in E_*$, u and v are incomparable w. r. t. $\succ$ and irreducible w. r. t. R_*.

Unfailing completion is refutationally complete for equational theories:

Theorem 4.39 *Let E be a set of equations, let $\succ$ be a reduction ordering that is total on ground terms. For any two terms s and t, let $\hat{s}$ and $\hat{t}$ be the terms obtained from s and t by replacing all variables by Skolem constants. Let $eq/2$, $true/0$ and $false/0$ be new operator symbols, such that $true$ and $false$ are smaller than all other terms. Let $E_0 = E \cup \{eq(\hat{s}, \hat{t}) \approx true,\ eq(x, x) \approx false\}$. If $E_0, \emptyset \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$ be a fair run of unfailing completion, then $s \approx_E t$ iff some $E_i \cup R_i$ contains $true \approx false$.*

Outlook:

Combine ordered resolution and unfailing completion to get a calculus for equational clauses:

compute inferences between (strictly) maximal literals as in ordered resolution,
compute overlaps between maximal sides of equations as in unfailing completion

$\Rightarrow$ Superposition calculus.