Confluence

Let \((A, \rightarrow)\) be a rewrite system.

\(b\) and \(c \in A\) are joinable, if there is an \(a\) such that \(b \rightarrow^* a \leftarrow^* c\).
Notation: \(b \downarrow c\).

The relation \(\rightarrow\) is called

- Church-Rosser, if \(b \leftrightarrow^* c\) implies \(b \downarrow c\).
- confluent, if \(b \leftarrow^* a \rightarrow^* c\) implies \(b \downarrow c\).
- locally confluent, if \(b \leftarrow a \rightarrow c\) implies \(b \downarrow c\).
- convergent, if it is confluent and terminating.
Confluence

For a rewrite system \((M, \rightarrow)\) consider a sequence of elements \(a_i\) that are pairwise connected by the symmetric closure, i.e., \(a_1 \leftrightarrow a_2 \leftrightarrow a_3 \ldots \leftrightarrow a_n\). We say that \(a_i\) is a peak in such a sequence, if actually \(a_{i-1} \leftarrow a_i \rightarrow a_{i+1}\).
Confluence

Theorem 1.11:
The following properties are equivalent:

(i) \( \rightarrow \) has the Church-Rosser property.

(ii) \( \rightarrow \) is confluent.
Confluence

Lemma 1.12: If $\rightarrow$ is confluent, then every element has at most one normal form.

Corollary 1.13: If $\rightarrow$ is normalizing and confluent, then every element $b$ has a unique normal form.

Proposition 1.14: If $\rightarrow$ is normalizing and confluent, then $b \leftrightarrow* c$ if and only if $b \Downarrow = c \Downarrow$. 


Confluence and Local Confluence

Theorem 1.15 ("Newman’s Lemma"): If a terminating relation $\rightarrow$ is locally confluent, then it is confluent.
Part 2: Propositional Logic

Propositional logic

- logic of truth values
- decidable (but $\mathbf{NP}$-complete)
- can be used to describe functions over a finite domain
- industry standard for many analysis/verification tasks
- growing importance for discrete optimization problems (Automated Reasoning II)
2.1 Syntax

- propositional variables
- logical connectives
  \( \Rightarrow \) Boolean connectives and constants
Propositional Variables

Let $\Sigma$ be a set of **propositional variables** also called the **signature** of the (propositional) logic.

We use letters $P$, $Q$, $R$, $S$, to denote propositional variables.
Propositional Formulas

PROP(Σ) is the set of propositional formulas over Σ inductively defined as follows:

\[
\begin{align*}
\phi, \psi & ::= \bot \quad \text{(falsum)} \\
& \quad \top \quad \text{(verum)} \\
& \quad P, \ P \in \Sigma \quad \text{(atomic formula)} \\
& \quad \neg \phi \quad \text{(negation)} \\
& \quad (\phi \land \psi) \quad \text{(conjunction)} \\
& \quad (\phi \lor \psi) \quad \text{(disjunction)} \\
& \quad (\phi \rightarrow \psi) \quad \text{(implication)} \\
& \quad (\phi \leftrightarrow \psi) \quad \text{(equivalence)}
\end{align*}
\]
Notational Conventions

As a notational convention we assume that $\neg$ binds strongest, so $\neg P \lor Q$ is actually a shorthand for $(\neg P) \lor Q$. For all other logical connectives we will explicitly put parenthesis when needed. From the semantics we will see that $\land$ and $\lor$ are associative and commutative. Therefore instead of $((P \land Q) \land R)$ we simply write $P \land Q \land R$.

Automated reasoning is very much formula manipulation. In order to precisely represent the manipulation of a formula, we introduce positions.
A position is a word over $\mathbb{N}$. The set of positions of a formula $\phi$ is inductively defined by

\[
\begin{align*}
pos(\phi) & := \{\epsilon\} \text{ if } \phi \in \{\top, \bot\} \text{ or } \phi \in \Sigma \\
pos(\neg\phi) & := \{\epsilon\} \cup \{1p \mid p \in \pos(\phi)\} \\
pos(\phi \circ \psi) & := \{\epsilon\} \cup \{1p \mid p \in \pos(\phi)\} \cup \{2p \mid p \in \pos(\psi)\}
\end{align*}
\]

where $\circ \in \{\land, \lor, \rightarrow, \leftrightarrow\}$.
The prefix order $\leq$ on positions is defined by $p \leq q$ if there is some $p'$ such that $pp' = q$.

Note that the prefix order is partial, e.g., the positions 12 and 21 are not comparable, they are “parallel”, see below.

By $<$ we denote the strict part of $\leq$, i.e., $p < q$ if $p \leq q$ but not $q \leq p$. By $\parallel$ we denote incomparable positions, i.e., $p \parallel q$ if neither $p \leq q$, nor $q \leq p$. Then we say that $p$ is above $q$ if $p \leq q$, $p$ is strictly above $q$ if $p < q$, and $p$ and $q$ are parallel if $p \parallel q$. 
Formula Manipulation

The size of a formula $\phi$ is given by the cardinality of $\text{pos}(\phi)$:

$$|\phi| := |\text{pos}(\phi)|.$$ 

The subformula of $\phi$ at position $p \in \text{pos}(\phi)$ is recursively defined by $\phi|_\epsilon := \phi$ and $(\phi_1 \circ \phi_2)|_ip := \phi_i|_p$ where $i \in \{1, 2\}$, $\circ \in \{\land, \lor, \rightarrow, \leftrightarrow\}$. 


Finally, the replacement of a subformula at position \( p \in \text{pos}(\phi) \) by a formula \( \psi \) is recursively defined by

\[
\begin{align*}
\phi[\psi]_e & := \psi \\
(\neg \phi)[\psi]_{1p} & := \neg(\phi[\psi]_p) \\
(\phi_1 \circ \phi_2)[\psi]_{1p} & := (\phi_1[\psi]_p \circ \phi_2) \\
(\phi_1 \circ \phi_2)[\psi]_{2p} & := (\phi_1 \circ \phi_2[\psi]_p)
\end{align*}
\]

where \( \circ \in \{\land, \lor, \rightarrow, \leftrightarrow\} \).
Example 2.1:
The set of positions for the formula $\phi = (A \land B) \rightarrow (A \lor B)$ is $\text{pos}(\phi) = \{\epsilon, 1, 11, 12, 2, 21, 22\}$. The subformula at position 22 is $B$, $\phi|_{22} = B$ and replacing this formula by $A \leftrightarrow B$ results in $\phi[A \leftrightarrow B]_{22} = (A \land B) \rightarrow (A \lor (A \leftrightarrow B))$. 
A further prerequisite for efficient formula manipulation is the polarity of a subformula $\psi$ of $\phi$. The polarity determines the number of “negations” starting from $\phi$ down to $\psi$. It is 1 for an even number along the path, $-1$ for an odd number and 0 if there is at least one equivalence connective along the path.
The **polarity** of a subformula $\psi$ of $\phi$ at position $p$, $i \in \{1, 2\}$ is recursively defined by

$$
\begin{align*}
\text{pol}(\phi, \epsilon) & := 1 \\
\text{pol}(\neg \phi, 1p) & := - \text{pol}(\phi, p) \\
\text{pol}(\phi_1 \circ \phi_2, ip) & := \text{pol}(\phi_i, p) \text{ if } \circ \in \{\land, \lor\} \\
\text{pol}(\phi_1 \rightarrow \phi_2, 1p) & := - \text{pol}(\phi_2, p) \\
\text{pol}(\phi_1 \rightarrow \phi_2, 2p) & := \text{pol}(\phi_2, p) \\
\text{pol}(\phi_1 \leftrightarrow \phi_2, ip) & := 0
\end{align*}
$$
Example 2.2:
We reuse the formula $\phi = (A \land B) \rightarrow (A \lor B)$ Then $\text{pol}(\phi, 1) = \text{pol}(\phi, 11) = -1$ and $\text{pol}(\phi, 2) = \text{pol}(\phi, 22) = 1$. For the formula $\phi' = (A \land B) \leftrightarrow (A \lor B)$ we get $\text{pol}(\phi', \epsilon) = 1$ and $\text{pol}(\phi', p) = 0$ for all other $p \in \text{pos}(\phi'), p \neq \epsilon$. 