2.5 Superposition for $\text{PROP}(\Sigma)$

Superposition for $\text{PROP}(\Sigma)$ is:

- resolution (Robinson 1965) +
- ordering restrictions (Bachmair & Ganzinger 1990) +
- abstract redundancy criterion (B&G 1990) +
- partial model construction (B & G 1990) +
- partial-model based inference restriction (Weidenbach)
Resolution for $\text{PROP}(\Sigma)$

A calculus is a set of inference and reduction rules for a given logic (here $\text{PROP}(\Sigma)$).

We only consider calculi operating on a set of clauses $N$. Inference rules add new clauses to $N$ whereas reduction rules remove clauses from $N$ or replace clauses by “simpler” ones.

We are only interested in unsatisfiability, i.e., the considered calculi test whether a clause set $N$ is unsatisfiable. So, in order to check validity of a formula $\phi$ we check unsatisfiability of the clauses generated from $\neg \phi$. 
Resolution for $\text{PROP}(\Sigma)$

For clauses we switch between the notation as a disjunction, e.g., $P \lor Q \lor P \lor \neg R$, and the notation as a multiset, e.g., \{ $P$, $Q$, $P$, $\neg R$ \}. This makes no difference as we consider $\lor$ in the context of clauses always modulo AC. Note that $\bot$, the empty disjunction, corresponds to $\emptyset$, the empty multiset.

For literals we write $L$, possibly with subscript. If $L = P$ then $\overline{L} = \neg P$ and if $L = \neg P$ then $\overline{L} = P$, so the bar flips the negation of a literal.

Clauses are typically denoted by letters $C$, $D$, possibly with subscript.
Resolution for $\text{PROP}(\Sigma)$

The resolution calculus consists of the inference rules resolution and factoring:

**Resolution**

\[
\frac{C_1 \lor P \quad C_2 \lor \neg P}{\frac{C_1 \lor C_2}{I}}
\]

**Factoring**

\[
\frac{C \lor L \lor L}{\frac{C \lor L}{I}}
\]

where $C_1$, $C_2$, $C$ always stand for clauses, all inference/reduction rules are applied with respect to AC of $\lor$. Given a clause set $N$ the schema above the inference bar is mapped to $N$ and the resulting clauses below the bar are then added to $N$. 
Resolution for $\text{PROP}(\Sigma)$

and the reduction rules subsumption and tautology deletion:

\[
\begin{align*}
\text{Subsumption} & \quad \text{Tautology Deletion} \\
\mathcal{R} & \quad \mathcal{R} \\
\frac{C_1 \quad C_2}{C_1} & \quad \frac{C \lor P \lor \neg P}{C} \\
\end{align*}
\]

where for subsumption we assume $C_1 \subseteq C_2$. Given a clause set $N$ the schema above the reduction bar is mapped to $N$ and the resulting clauses below the bar replace the clauses above the bar in $N$.

Clauses that can be removed are called redundant.
Resolution for \( \text{PROP}(\Sigma) \)

So, if we consider clause sets \( N \) as states, \( \sqcup \) is disjoint union, we get the rules

\[
\text{Resolution} \quad (N \sqcup \{C_1 \lor P, C_2 \lor \neg P\}) \Rightarrow (N \cup \{C_1 \lor P, C_2 \lor \neg P\} \cup \{C_1 \lor C_2\})
\]

\[
\text{Factoring} \quad (N \sqcup \{C \lor L \lor L\}) \Rightarrow (N \cup \{C \lor L \lor L\} \cup \{C \lor L\})
\]
Resolution for \( \text{PROP}(\Sigma) \)

**Subsumption**  \( (N \uplus \{C_1, C_2\}) \Rightarrow (N \cup \{C_1\}) \)

provided \( C_1 \subseteq C_2 \)

**Tautology**  \( (N \uplus \{C \lor P \lor \neg P\}) \Rightarrow (N) \)

**Deletion**

We need more structure than just \((N)\) in order to define a useful rewrite system. We fix this later on.
Resolution for $PROP(\Sigma)$

Theorem 2.11:
The resolution calculus is sound and complete:

\[ N \text{ is unsatisfiable iff } N \Rightarrow^* \{ \bot \} \]

Proof:
Will be a consequence of soundness and completeness of superposition. \qed
Ordering restrictions

Let \( \prec \) be a total ordering on \( \Sigma \).

We lift \( \prec \) to a total ordering on literals by \( \prec \subseteq \prec_L \) and \( P \prec_L \neg P \) and \( \neg P \prec_L Q \) for all \( P \prec Q \).

We further lift \( \prec_L \) to a total ordering on clauses \( \prec_C \) by considering the multiset extension of \( \prec_L \) for clauses.

Eventually, we overload \( \prec \) with \( \prec_L \) and \( \prec_C \).

We define \( N^{\prec_C} = \{ D \in N \mid D \prec C \} \).
Ordering restrictions

Eventually we will restrict inferences to maximal literals with respect to $\prec$. 

Abstract Redundancy

A clause $C$ is redundant with respect to a clause set $N$ if $N \prec C \models C$.

Tautologies are redundant. Subsumed clauses are redundant if $\subseteq$ is strict.

Remark: Note that for finite $N$, $N \prec C \models C$ can be decided for $PROP(\Sigma)$ but is as hard as testing unsatisfiability for a clause set $N$. 
Partial Model Construction

Given a clause set $N$ and an ordering $\prec$ we can construct a (partial) model $N_{\mathcal{I}}$ for $N$ as follows:

$$N_C := \bigcup_{D \prec C} \delta_D$$

$$\delta_D := \begin{cases} 
\{P\} & \text{if } D = D' \lor P \text{ and } P \text{ maximal and } N_D \not\vdash D \\
\emptyset & \text{otherwise}
\end{cases}$$

$$N_{\mathcal{I}} := \bigcup_{C \in N} \delta_C$$
Superposition

The superposition calculus consists of the inference rules superposition left and factoring:

**Superposition**

\[(N \cup \{ C_1 \lor P, C_2 \lor \neg P \}) \Rightarrow (N \cup \{ C_1 \lor P, C_2 \lor \neg P, C_1 \lor C_2 \})\]

where \(P\) is strictly maximal in \(C_1 \lor P\) and \(\neg P\) is maximal in \(C_2 \lor \neg P\)

**Factoring**

\[(N \cup \{ C \lor P \lor P \}) \Rightarrow (N \cup \{ C \lor P \lor P, C \lor P \})\]

where \(P\) is maximal in \(C \lor P \lor P\)
Superposition

examples for specific redundancy rules are

**Subsumption**  \((N \uplus \{C_1, C_2\}) \Rightarrow (N \cup \{C_1\})\)

provided \(C_1 \subset C_2\)

**Tautology Deletion**  \((N \uplus \{C \vee P \vee \neg P\}) \Rightarrow (N)\)

**Subsumption Resolution**  \((N \uplus \{C_1 \vee L, C_2 \vee \bar{L}\}) \Rightarrow (N \cup \{C_1 \vee L, C_2\})\)

where \(C_1 \subseteq C_2\)
Theorem 2.12:
If from a clause set $N$ all possible superposition inferences are redundant and $\bot \notin N$ then $N$ is satisfiable and $N_I \models N$. 

Superposition