Rule-Based Naive Standard Unification

\[
\begin{align*}
t & \doteq t, E \quad \Rightarrow_{SU} \quad E \\
f(s_1, \ldots, s_n) & \doteq f(t_1, \ldots, t_n), E \quad \Rightarrow_{SU} \quad s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) & \doteq g(\ldots), E \quad \Rightarrow_{SU} \quad \bot \\
x & \doteq t, E \quad \Rightarrow_{SU} \quad x \doteq t, E\{x \mapsto t\} \\
& \quad \text{if } x \in \text{var}(E), x \not\in \text{var}(t) \\
x & \doteq t, E \quad \Rightarrow_{SU} \quad \bot \\
& \quad \text{if } x \neq t, x \in \text{var}(t) \\
t & \doteq x, E \quad \Rightarrow_{SU} \quad x \doteq t, E \\
& \quad \text{if } t \not\in X
\end{align*}
\]
If \( E = x_1 = u_1, \ldots, x_k = u_k \), with \( x_i \) pairwise distinct, \( x_i \not\in \text{var}(u_j) \), then \( E \) is called an (equational problem in) solved form representing the solution \( \sigma_E = \{x_1 \mapsto u_1, \ldots, x_k \mapsto u_k\} \).

Proposition 3.22:
If \( E \) is a solved form then \( \sigma_E \) is an mgu of \( E \).
SU: Main Properties

Theorem 3.23:

1. If $E \Rightarrow_{SU} E'$ then $\sigma$ is a unifier of $E$ iff $\sigma$ is a unifier of $E'$

2. If $E \Rightarrow^*_{SU} \bot$ then $E$ is not unifiable.

3. If $E \Rightarrow^*_{SU} E'$ with $E'$ in solved form, then $\sigma_{E'}$ is an mgu of $E$.

Proof:

(1) We have to show this for each of the rules. Let’s treat the case for the 4th rule here. Suppose $\sigma$ is a unifier of $x = t$, that is, $x\sigma = t\sigma$. Thus, $\sigma \circ \{ x \mapsto t \} = \sigma[ x \mapsto t\sigma] = \sigma[ x \mapsto x\sigma] = \sigma$. Therefore, for any equation $u = v$ in $E$: $u\sigma = v\sigma$, iff $u\{ x \mapsto t \}\sigma = v\{ x \mapsto t \}\sigma$. (2) and (3) follow by induction from (1) using Proposition 3.22.  

\[ \square \]
Main Unification Theorem

Theorem 3.24:
$E$ is unifiable if and only if there is a most general unifier $\sigma$ of $E$, such that $\sigma$ is idempotent and $\text{dom}(\sigma) \cup \text{codom}(\sigma) \subseteq \text{var}(E)$. 
Rule-Based Polynomial Unification

Problem: using $\Rightarrow_{SU}$, an exponential growth of terms is possible.

The following unification algorithm avoids this problem, at least if the final solved form is represented as a DAG.
Rule-Based Polynomial Unification

\[ t \doteq t, E \quad \Rightarrow_{PU} \quad E \]

\[ f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E \quad \Rightarrow_{PU} \quad s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \]

\[ f(\ldots) \doteq g(\ldots), E \quad \Rightarrow_{PU} \quad \perp \]

\[ x \doteq y, E \quad \Rightarrow_{PU} \quad x \doteq y, E \{ x \mapsto y \} \]

\[ \text{if } x \in \text{var}(E), x \neq y \]

\[ x_1 \doteq t_1, \ldots, x_n \doteq t_n, E \quad \Rightarrow_{PU} \quad \perp \]

\[ \text{if there are positions } p_i \text{ with } \]

\[ t_i/p_i = x_{i+1}, t_n/p_n = x_1 \]

\[ \text{and some } p_i \neq \epsilon \]
Rule-Based Polynomial Unification

\[ x \doteq t, E \Rightarrow_{PU} \bot \]

if \( x \neq t, x \in \text{var}(t) \)

\[ t \doteq x, E \Rightarrow_{PU} x \doteq t, E \]

if \( t \not\in X \)

\[ x \doteq t, x \doteq s, E \Rightarrow_{PU} x \doteq t, t \doteq s, E \]

if \( t, s \not\in X \) and \( |t| \leq |s| \)
Properties of PU

Theorem 3.25:

1. If $E \Rightarrow_{PU} E'$ then $\sigma$ is a unifier of $E$ iff $\sigma$ is a unifier of $E'$

2. If $E \Rightarrow_{PU}^{*} \bot$ then $E$ is not unifiable.

3. If $E \Rightarrow_{PU}^{*} E'$ with $E'$ in solved form, then $\sigma_{E'}$ is an mgu of $E$.

Note: The solved form of $\Rightarrow_{PU}$ is different form the solved form obtained from $\Rightarrow_{SU}$. In order to obtain the unifier $\sigma_{E'}$, we have to sort the list of equality problems $x_i \doteq t_i$ in such a way that $x_i$ does not occur in $t_j$ for $j < i$, and then we have to compose the substitutions $\{x_1 \mapsto t_1\} \circ \cdots \circ \{x_k \mapsto t_k\}$. 
Lemma 3.26:
Let $C$ and $D$ be variable-disjoint clauses. If

\[
\begin{array}{c}
D \\
\sigma \\
\hline
D\sigma \\
\end{array}
\quad
\begin{array}{c}
C \\
\rho \\
\hline
C\rho \\
\end{array}
\]

then there exists a substitution $\tau$ such that

\[
\begin{array}{c}
D \\
\hline
C'' \\
\end{array}
\quad
\begin{array}{c}
C' \\
\tau \\
\hline
C''\tau \\
\end{array}
\]

[propositional resolution]
[general resolution]
Lifting Lemma

An analogous lifting lemma holds for factorization.