Problem 1 (Unification) (6 points)

For each of the following unification problems, compute either an mgu or show that it is not unifiable:

\[
E_1 = \{ f(g(x), x) =? f(y, h(y)) \} \\
E_2 = \{ h(a, z, b) =? h(x, x, y) \} \\
E_3 = \{ g(x, f(x)) =? g(y, z), g(x', x') =? g(y, f(z')) \} 
\]

Problem 2 (Well-founded Orderings) (6 points)

Let \((A, >)\) be a well-founded partial ordering, let \(f : A \rightarrow A\) be a monotone function (that is, \(x > y\) implies \(f(x) > f(y)\) for all elements \(x, y \in A\)). Prove:

If \(x \geq f(x)\) for all \(x \in A\), then \(x = f(x)\) for all \(x \in A\).

Problem 3 (Reduction Orderings) (8 points)

The proper subterm relation \(\triangleright\) is defined by

\[ s \triangleright t \text{ if and only if there is a } p \in \text{Pos}(s) \text{ such that } p \neq \varepsilon \text{ and } s/p = t. \]

Is the proper subterm relation a reduction ordering? Give a proof or a counterexample.

Problem 4 (Multisets) (4 + 4 = 8 points)

Let \(N = \{M_1, M_2, M_3, M_4, M_5\}\) be a set of multisets of multisets:

\[
M_1 = \{\{a_4\}, \{a_4\}, \{a_1\}, \{a_1\}\} \\
M_2 = \{\{a_2\}, \{a_1\}, \{a_1\}\} \\
M_3 = \{\{a_3, a_1\}\} \\
M_4 = \{\{a_4, a_3\}, \{a_3, a_2\}, \{a_2, a_1, a_1\}\} \\
M_5 = \{\{a_2\}, \{a_1, a_1\}, \emptyset\} 
\]

Part (a)

Let the ordering \(\triangleright\) be defined by \(a_4 \triangleright a_3 \triangleright a_2 \triangleright a_1\), let \(\triangleright_m\) be the multiset extension of \(\triangleright\), and let \(\triangleright_{mm}\) be the multiset extension of \(\triangleright_m\). Sort the elements of \(N\) with respect to \(\triangleright_{mm}\).

Part (b)

Find another total ordering \(\triangleright'\) on \(\{a_1, a_2, a_3, a_4\}\) such that \(M_3\) is maximal and \(M_1\) is minimal in \(N\) with respect to \(\triangleright'_{mm}\), where \(\triangleright'_{mm}\) is the twofold multiset extension of \(\triangleright'\).
Problem 5 (Confluence) (10 points)

For a term rewrite system \( R \), we define LSymb\((R)\) as the set of all function symbols occurring in the left-hand sides of rules in \( R \). More formally,

\[
\text{LSymb}(R) = \bigcup_{l \rightarrow r \in R} \text{Symb}(l),
\]

where \( \text{Symb}(x) = \emptyset \) and \( \text{Symb}(f(t_1, \ldots, t_n)) = \{f\} \cup \bigcup_{i=1}^{n} \text{Symb}(t_i) \).

Prove: If \( R_1 \) and \( R_2 \) are confluent term rewrite systems, such that \( R_1 \cup R_2 \) is terminating, and \( \text{LSymb}(R_1) \cap \text{LSymb}(R_2) = \emptyset \), then \( R_1 \cup R_2 \) is confluent.