Let \((A, >_A)\) and \((B, >_B)\) be partial orderings. A mapping \(\varphi : A \rightarrow B\) is called monotone, if \(x >_A y\) implies \(\varphi(x) >_B \varphi(y)\) for all \(x, y \in A\).

Lemma:
If \(\varphi : A \rightarrow B\) is a monotone mapping from \((A, >_A)\) to \((B, >_B)\) and \((B, >_B)\) is well-founded, then \((A, >_A)\) is well-founded.
Proving Termination: Lexicographic Product

Let \((A, >_A)\) and \((B, >_B)\) be partial orderings. The lexicographic product \(>_A \times B\) on \(A \times B\) is defined by

\[
(x, y) >_A \times B (x', y') \iff (x >_A x') \lor (x = x' \land y >_B y').
\]

Lemma:
The lexicographic product of two partial orderings is a partial ordering.
Lemma:
The lexicographic product of two well-founded partial orderings is a well-founded partial ordering.

Proof:
Assume that there is an infinite decreasing chain

\((a_0, b_0) >_{A \times B} (a_1, b_1) >_{A \times B} \ldots\)

This implies \(a_0 \geq_A a_1 \geq_A \ldots\).
Since \(>_A\) is well-founded, this chain can only contain finitely many strict steps \(a_i >_A a_{i+1}\).
Hence there is a \(k\) such that \(a_i = a_{i+1}\) for all \(i \geq k\). But then \(b_i >_B b_{i+1}\) for all \(i \geq k\), contradicting the well-foundedness of \(>_B\).
Proving Termination: Lexicographic Product

Lemma:
The lexicographic product of two strict total orderings is a strict total ordering.

Proof:
by case analysis.
The lexicographic product $\preceq_{\text{lex}}^n$ of partial orderings $(A_i, >_i)$ with $1 \leq i \leq n$ can be defined analogously for $n$-tuples with $n > 2$.

The resulting relation is again a partial ordering; it is well-founded if the orderings $(A_i, >_i)$ are well-founded, and it is total if the orderings $(A_i, >_i)$ are total.
Proving Termination: Lexicographic Product

Note: Given an ordering \((A, >_A)\), one can define a lexicographic ordering \(>_\text{Lex} \) on \(A^* = \bigcup_{i \geq 0} A^i\) by

\[
w >_{\text{Lex}} w' \iff (w = w'v \land |v| > 0) \\
\lor (w = uxv \land w' = ux'v' \land x >_A x').
\]

However, this ordering is not well-founded!
Proving Termination: Lexicographic Product

To get a well-founded ordering on $A^*$, one has to compare the length of tuples first ("length/lexicographic combination"):\[
\begin{align*}
w >^{*_{lex}} w' \iff & \ (|w| > |w'|) \\
& \lor (|w| = |w'| = n \land w >^n_{lex} w')
\end{align*}
\]

where $>^n_{lex}$ is the lexicographic ordering on $n$-tuples.
A multiset $M$ over $A$ is a function $M : A \rightarrow \mathbb{N}$.

Intuitively, a multiset is a set with (finitely often) repeated elements; $M(x)$ is the number of copies of $x$ in $M$.

We use similar notation as for sets; for instance we write $\{a, c, c\}$ for the multiset $\{a \mapsto 1, b \mapsto 0, c \mapsto 2\}$. 
A multiset $M$ is called finite, if $\{ x \in A \mid M(x) > 0 \}$ is finite.

$\mathcal{M}(A)$ denotes the set of all finite multisets over $A$.

From now on we will consider only finite multisets.
Notations:

**Element:**  \( x \in M \iff M(x) > 0 \)

**Submultiset:**  \( M \subseteq N \iff \text{for all } x \in A: \ M(x) \leq N(x) \)

**Union:**  \( (M \cup N)(x) = M(x) + N(x) \)

**Difference:**  \( (M \setminus N)(x) = M(x) \div N(x) \)

**Intersection:**  \( (M \cap N)(x) = \min \{M(x), N(x)\} \)

where \( m \div n = m - n \) if \( m \geq n \), and \( m \div n = 0 \) otherwise.
Multiset extension:

Let \((A, >)\) be a partial ordering. We define an ordering \(>_{mul}\) over \(\mathcal{M}(A)\) as follows:

\[
M >_{mul} N \quad \text{iff} \quad \text{there exist } X, Y \in \mathcal{M}(A) \text{ such that }
\]
\[
\emptyset \neq X \subseteq M \text{ and }
\]
\[
N = (M \setminus X) \cup Y \text{ and }
\]
\[
\forall y \in Y \exists x \in X : x > y.
\]
Lemma:
The multiset extension $\succ_{mul}$ of a partial ordering $\succ$ is a partial ordering.

Proof:
Baader and Nipkow, page 22/23.
Lemma ("König’s Lemma"): A finitely branching tree is infinite, if and only if it contains an infinite path.

Proof:
“if” : trivial.
“only if” : by well-founded induction over the subtree relation.
Theorem:
The multiset extension of a partial ordering $>$ is well-founded if and only if $>$ is well-founded.

Proof:
Lemma: \( M >_{\text{mul}} N \) if and only if \( M \neq N \) and for every \( n \in N \setminus M \) there is an \( m \in M \setminus N \) such that \( m > n \).

Proof:
Baader and Nipkow, page 24/25.
Corollary: If the ordering $>$ is total, then its multiset extension $>_{mul}$ is total.

Proof:
Let $>$ be total.
If the multisets $M$ and $N$ are different, then there exists a greatest element $m \in A$ such that $M(m) \neq N(m)$.
W.o.l.o.g, let $M(m) > N(m)$, hence $m \in M \setminus N$.
Then for every $n \in N \setminus M$ we have $m > n$, hence $M >_{mul} N$. 
3 Rewrite Systems
Let $E$ be a set of equations.

The rewrite relation $\rightarrow_E \subseteq T_\Sigma(X) \times T_\Sigma(X)$ is defined by

$$s \rightarrow_E t \text{ iff there exist } (l \approx r) \in E, \ p \in \text{Pos}(s),$$
and $\sigma : X \rightarrow T_\Sigma(X)$,

such that $s/p = l\sigma$ and $t = s[r\sigma]_p$. 
Rewrite Relations

An equation $l \approx r$ is also called a rewrite rule, if $l$ is not a variable and $\text{Var}(l) \supseteq \text{Var}(r)$.

Notation: $l \rightarrow r$.

A set of rewrite rules is called a term rewrite system (TRS).
We say that a set of equations $E$ or a TRS $R$ is terminating, if the rewrite relation $\rightarrow_E$ or $\rightarrow_R$ has this property.

(Analogously for other properties of abstract reduction systems).