Let $E$ be a set of equations.

The rewrite relation $\rightarrow_E \subseteq T_\Sigma(X) \times T_\Sigma(X)$ is defined by

$s \rightarrow_E t$ iff there exist $(l \approx r) \in E$, $p \in \text{Pos}(s)$, and $\sigma : X \rightarrow T_\Sigma(X)$, such that $s/p = l\sigma$ and $t = s[r\sigma]_p$.

An instance of the lhs (left-hand side) of an equation is called a **redex** (reducible expression). **Contracting** a redex means replacing it with the corresponding instance of the rhs (right-hand side) of the rule.
An equation $l \approx r$ is also called a rewrite rule, if $l$ is not a variable and $\text{Var}(l) \supseteq \text{Var}(r)$.

Notation: $l \rightarrow r$.

A set of rewrite rules is called a term rewrite system (TRS).
We say that a set of equations $E$ or a TRS $R$ is terminating, if the rewrite relation $\rightarrow_E$ or $\rightarrow_R$ has this property.

(Analogously for other properties of abstract reduction systems).

Note: If $E$ is terminating, then it is a TRS.
E-Algebras

Let $E$ be a set of closed equations. A $\Sigma$-algebra $\mathcal{A}$ is called an $E$-algebra, if $\mathcal{A} \models \forall \vec{x}(s \approx t)$ for all $\forall \vec{x}(s \approx t) \in E$.

If $E \models \forall \vec{x}(s \approx t)$ (i.e., $\forall \vec{x}(s \approx t)$ is valid in all $E$-algebras), we write this also as $s \approx_E t$.

Goal:
Use the rewrite relation $\rightarrow_E$ to express the semantic consequence relation syntactically:

\[ s \approx_E t \text{ if and only if } s \leftrightarrow_E^* t. \]
Let $E$ be a set of equations over $T_{\Sigma}(X)$. The following inference system allows to derive consequences of $E$: 

E-Algebras
E-Algebras

\[
E \vdash t \approx t \\
E \vdash t \approx t' \\
E \vdash t' \approx t
\]

(Reflexivity)

\[
E \vdash t \approx t' \\
E \vdash t' \approx t'' \\
E \vdash t \approx t''
\]

(Symmetry)

(Transitivity)

\[
E \vdash t_1 \approx t'_1 \ldots E \vdash t_n \approx t'_n \\
E \vdash f(t_1, \ldots, t_n) \approx f(t'_1, \ldots, t'_n)
\]

(Congruence)

\[
E \vdash t\sigma \approx t'\sigma
\]

(Instance)

if \((t \approx t') \in E\) and \(\sigma : X \rightarrow T_\Sigma(X)\)
Lemma:
The following properties are equivalent:

(i) \( s \leftrightarrow^*_E t \)

(ii) \( E \vdash s \approx t \) is derivable.

Proof:
(i)\(\implies\)(ii): \( s \leftrightarrow_E t \) implies \( E \vdash s \approx t \) by induction on the depth of the position where the rewrite rule is applied; then \( s \leftrightarrow^*_E t \) implies \( E \vdash s \approx t \) by induction on the number of rewrite steps in \( s \leftrightarrow^*_E t \).

(ii)\(\implies\)(i): By induction on the size of the derivation for \( E \vdash s \approx t \).
Constructing a quotient algebra:

Let $X$ be a set of variables.

For $t \in T_\Sigma(X)$ let $[t] = \{ t' \in T_\Sigma(X) \mid E \vdash t \approx t' \}$ be the congruence class of $t$.

Define a $\Sigma$-algebra $T_\Sigma(X)/E$ (abbreviated by $T$) as follows:

$U_T = \{ [t] \mid t \in T_\Sigma(X) \}$.

$f_T([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$ for $f/n \in \Omega$. 
E-Algebras

Lemma:
\( f_T \) is well-defined:
If \([t_i] = [t'_i]\), then \([f(t_1, \ldots, t_n)] = [f(t'_1, \ldots, t'_n)]\).

Proof:
Follows directly from the Congruence rule for \( \vdash \).
Lemma:
\[ T = T_{\Sigma}(X)/E \] is an \( E \)-algebra.

Proof:
Let \( \forall \overline{x}(s \approx t) \) be an equation in \( E \); let \( \alpha \) be an arbitrary assignment.

We have to show that \( T(\alpha)(\forall \overline{x}(s \approx t)) = 1 \), or equivalently, that \( T(\beta)(s) = T(\beta)(t) \) for all \( \beta = \alpha[ x_i \mapsto [t_i] \mid i \in I ] \) with \([t_i] \in U_T\).

Let \( \sigma = \{ x_i \mapsto t_i \mid i \in I \} \), then \( s\sigma \in T(\beta)(s) \) and \( t\sigma \in T(\beta)(t) \).

By the Instance rule, \( E \vdash s\sigma \approx t\sigma \) is derivable, hence \( T(\beta)(s) = [s\sigma] = [t\sigma] = T(\beta)(t) \).
Lemma:
Let $X$ be a countably infinite set of variables; let $s, t \in T_\Sigma(X)$. If $T_\Sigma(X)/E \models \forall \bar{x}(s \approx t)$, then $E \vdash s \approx t$ is derivable.

Proof:
Assume that $T \models \forall \bar{x}(s \approx t)$, i.e., $T(\alpha)(\forall \bar{x}(s \approx t)) = 1$. Consequently, $T(\beta)(s) = T(\beta)(t)$ for all $\beta = \alpha[x_i \mapsto [t_i] \mid i \in I]$ with $[t_i] \in U_T$.

Choose $t_i = x_i$, then $[s] = T(\beta)(s) = T(\beta)(t) = [t]$, so $E \vdash s \approx t$ is derivable by definition of $T$. 

E-Algebras
Theorem ("Birkhoff’s Theorem"):
Let $X$ be a countably infinite set of variables, let $E$ be a set of (universally quantified) equations. Then the following properties are equivalent for all $s, t \in T_{\Sigma}(X)$:

(i) $s \leftrightarrow^*_E t$.

(ii) $E \vdash s \approx t$ is derivable.

(iii) $s \approx_E t$, i.e., $E \models \forall \vec{x}(s \approx t)$.

(iv) $T_{\Sigma}(X)/E \models \forall \vec{x}(s \approx t)$. 

E-Algebras
Proof:

(i)⇔(ii): See above (slide 7).

(ii)⇒(iii): By induction on the size of the derivation for $E \vdash s \approx t$.

(iii)⇒(iv): Obvious, since $\mathcal{T} = \mathcal{T}_E(X)$ is an $E$-algebra.

(iv)⇒(ii): See above (slide 11).
Universal Algebra

$T_\Sigma(X)/E = T_\Sigma(X)/\sim_E = T_\Sigma(X)/\leftrightarrow^*_E$ is called the free $E$-algebra with generating set $X/\sim_E = \{ [x] \mid x \in X \}$:

Every mapping $\varphi : X/\sim_E \to B$ for some $E$-algebra $B$ can be extended to a homomorphism $\hat{\varphi} : T_\Sigma(X)/E \to B$.

$T_\Sigma(\emptyset)/E = T_\Sigma(\emptyset)/\sim_E = T_\Sigma(\emptyset)/\leftrightarrow^*_E$ is called the initial $E$-algebra.
Universal Algebra

\[ \approx_E = \{ (s, t) \mid E \models s \approx t \} \]

is called the **equational theory** of \( E \).

\[ \approx^I_E = \{ (s, t) \mid T_\Sigma(\emptyset)/E \models s \approx t \} \]

is called the **inductive theory** of \( E \).

**Example:**

Let \( E = \{ \forall x(x + 0 \approx x), \ \forall x \forall y(x + s(y) \approx s(x + y)) \} \).

Then \( x + y \approx^I_E y + x \), but \( x + y \not\approx_E y + x \).
Corollary:
If $E$ is convergent (i.e., terminating and confluent), then $s \approx_E t$ if and only if $s \leftrightarrow_E^* t$ if and only if $s \downarrow_E = t \downarrow_E$.

Corollary:
If $E$ is finite and convergent, then $\approx_E$ is decidable.

Reminder:
If $E$ is terminating, then it is confluent if and only if it is locally confluent.
Rewrite Relations

Problems:

Show local confluence of $E$.

Show termination of $E$.

Transform $E$ into an equivalent set of equations that is locally confluent and terminating.
Rewrite Relations

Showing local confluence (Sketch):

Problem: If $t_1 \leftarrow_E t_0 \rightarrow_E t_2$, does there exist a term $s$ such that $t_1 \rightarrow^*_E s \leftarrow^*_E t_2$?

If the two rewrite steps happen in different subtrees (disjoint redexes): yes.

If the two rewrite steps happen below each other (overlap at or below a variable position): yes.

If the left-hand sides of the two rules overlap at a non-variable position: needs further investigation.
Showing local confluence (Sketch):

Question: Are there rewrite rules $l_1 \rightarrow r_1$ and $l_2 \rightarrow r_2$ such that $l_1$ and some subterm $l_2/p$ have a common instance $l_1\sigma_1 = (l_2/p)\sigma_2$?

Without loss of generality: assume that the two rewrite rules do not have common variables.
Then: Only a single substitution required: $l_1\sigma = (l_2/p)\sigma$?

Further questions:
For which substitutions $\sigma$ can this happen?
If there are infinitely many substitutions, can we describe them finitely?