Simplification Orderings

The proper subterm ordering $\rhd$ is defined by $s \rhd t$ if and only if $s/p = t$ for some position $p \neq \varepsilon$ of s.

Simplification Orderings

A rewrite ordering $>$ over $\mathrm{T}_\Sigma(X)$ is called simplification ordering, if it has the subterm property:
$s \rhd t$ implies $s > t$ for all $s, t \in \mathrm{T}_\Sigma(X)$.

Example:

Let R_{emb} be the rewrite system
$R_{\text{emb}} = \{\, f(x_1, \ldots, x_n) \to x_i \mid f/n \in \Omega,\ n \geq 1,\ 1 \leq i \leq n \,\}$.

Define $\rhd_{\text{emb}} = \to^{+}_{R_{\text{emb}}}$ and $\unrhd_{\text{emb}} = \to^{*}_{R_{\text{emb}}}$
("homeomorphic embedding relation").

$\rhd_{\text{emb}}$ is a simplification ordering.

Simplification Orderings

Lemma:

If $>$ is a simplification ordering, then $s \rhd_{\text{emb}} t$ implies $s > t$ and $s \unrhd_{\text{emb}} t$ implies $s \geq t$.

Proof:

Since $>$ is transitive and $\geq$ is transitive and reflexive, it suffices to show that $s \rightarrow_{R_{\text{emb}}} t$ implies $s > t$.

By definition, $s \rightarrow_{R_{\text{emb}}} t$ if and only if $s = s[l\sigma]$ and $t = s[r\sigma]$ for some rule $l \rightarrow r \in R_{\text{emb}}$.

Obviously, $l \rhd r$ for all rules in R_{emb}, hence $l > r$.

Since $>$ is a rewrite relation, $s = s[l\sigma] > s[r\sigma] = t$.

Simplification Orderings

Goal:

Show that every simplification ordering is well-founded (and therefore a reduction ordering).

Note: This works only for finite signatures!

To fix this for infinite signatures, the definition of simplification orderings and the definition of embedding have to be modified.

Kruskal's Theorem

A (usually not strict) partial ordering $\succeq$ on a set A is called well-partial-ordering (wpo), if for every infinite sequence $a_1, a_2, a_3, \ldots$ there are indices $i < j$ such that $a_i \preceq a_j$.

Terminology:

An infinite sequence $a_1, a_2, a_3, \ldots$ is called good, if there exist $i < j$ such that $a_i \preceq a_j$; otherwise it is called bad.

Therefore: $\succeq$ is a wpo iff every infinite sequence is good.

Kruskal's Theorem

Lemma:
If $\succeq$ is a wpo, then every infinite sequence $a_1, a_2, a_3, \ldots$ has an infinite ascending subsequence $a_{i_1} \preceq a_{i_2} \preceq a_{i_3} \preceq \ldots$, where $i_1 < i_2 < i_3 < \ldots$.

Proof:
Let $a_1, a_2, a_3, \ldots$ be an infinite sequence. We call an index $m \geq 1$ terminal, if there is no $n > m$ such that $a_m \preceq a_n$.
There are only finitely many terminal indices $m_1, m_2, m_3, \ldots$;
otherwise the sequence $a_{m_1}, a_{m_2}, a_{m_3}, \ldots$ would be bad.
Choose $p > 1$ such that all $m \geq p$ are not terminal; define $i_1 = p$; define recursively i_{j+1} such that $i_{j+1} > i_j$ and $a_{i_{j+1}} \succeq a_{i_j}$.

Kruskal's Theorem

Lemma:

If $\succeq_1, \ldots, \succeq_n$ are wpo's on $A_1, \ldots, A_n$, then $\succeq$ defined by

$$(a_1, \ldots, a_n) \succeq (a'_1, \ldots, a'_n) \text{ iff } a_i \succeq_i a'_i \text{ for all } i$$

is a wpo on $A_1 \times \cdots \times A_n$.

Proof:

The case $n = 1$ is trivial.
Otherwise let $(a_1^{(1)}, \ldots, a_n^{(1)}), (a_1^{(2)}, \ldots, a_n^{(2)}), \ldots$ be an infinite sequence. By the previous lemma, there are infinitely many indices $i_1 < i_2 < i_3 < \ldots$ such that $a_n^{(i_1)} \preceq a_n^{(i_2)} \preceq a_n^{(i_3)} \preceq \ldots$.
By induction on n, there are $k < l$ such that $a_1^{(i_k)} \preceq a_1^{(i_l)} \wedge \cdots \wedge a_{n-1}^{(i_k)} \preceq a_{n-1}^{(i_l)}$. Therefore $(a_1^{(i_k)}, \ldots, a_n^{(i_k)}) \preceq (a_1^{(i_l)}, \ldots, a_n^{(i_l)})$.

Kruskal's Theorem

Theorem ("Kruskal's Theorem"):

Let Σ be a finite signature, let X be a finite set of variables.

Then $\trianglerighteq_{\mathrm{emb}}$ is a wpo on $\mathrm{T}_\Sigma(X)$.

Proof:

Baader and Nipkow, page 114/115.

Simplification Orderings

Theorem (Dershowitz):
If Σ is a finite signature, then every simplification ordering $>$ on $\mathrm{T}_\Sigma(X)$ is well-founded (and therefore a reduction ordering).

Proof:
Suppose that $t_1 > t_2 > t_3 > \ldots$ is an infinite decreasing chain.

First assume that there is an $x \in \mathrm{Var}(t_{i+1}) \setminus \mathrm{Var}(t_i)$.
Let $\sigma = \{x \mapsto t_i\}$, then $t_{i+1}\sigma \trianglerighteq x\sigma = t_i$ and therefore $t_i = t_i\sigma > t_{i+1}\sigma \geq t_i$, contradicting reflexivity.

Consequently, $\mathrm{Var}(t_i) \supseteq \mathrm{Var}(t_{i+1})$ and $t_i \in \mathrm{T}_\Sigma(V)$ for all i, where V is the finite set $\mathrm{Var}(t_1)$. By Kruskal's Theorem, there are $i < j$ with $t_i \trianglelefteq_{\mathrm{emb}} t_j$. Hence $t_i \leq t_j$, contradicting $t_i > t_j$.

Simplification Orderings

There are reduction orderings that are not simplification orderings and terminating TRSs that are not contained in any simplification ordering.

Example:

Let $R = \{f(f(x)) \to f(g(f(x)))\}$.

R terminates and $\to_R^+$ is therefore a reduction ordering.

Assume that $\to_R$ were contained in a simplification ordering $\succ$. Then $f(f(x)) \to_R f(g(f(x)))$ implies $f(f(x)) \succ f(g(f(x)))$, and $f(g(f(x))) \trianglerighteq_{\text{emb}} f(f(x))$ implies $f(g(f(x))) \succeq f(f(x))$, hence $f(f(x)) \succ f(f(x))$.

Recursive Path Orderings

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $>$ be a strict partial ordering ("precedence") on Ω.

The lexicographic path ordering $>_{\mathrm{lpo}}$ on $\mathrm{T}_\Sigma(X)$ induced by $>$ is defined by: $s >_{\mathrm{lpo}} t$ iff

(1) $t \in \mathrm{Var}(s)$ and $t \neq s$, or

(2) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and

- (a) $s_i \geq_{\mathrm{lpo}} t$ for some i, or
- (b) $f > g$ and $s >_{\mathrm{lpo}} t_j$ for all j, or
- (c) $f = g$, $s >_{\mathrm{lpo}} t_j$ for all j, and $(s_1, \dots, s_m)\ (>_{\mathrm{lpo}})_{\mathrm{lex}}\ (t_1, \dots, t_n)$.

Recursive Path Orderings

Lemma:

$s >_{\mathsf{lpo}} t$ implies $\mathsf{Var}(s) \supseteq \mathsf{Var}(t)$.

Proof:

By induction on $|s| + |t|$ and case analysis.

Recursive Path Orderings

Theorem:

$>_{\mathsf{lpo}}$ is a simplification ordering on $\mathsf{T}_{\Sigma}(X)$.

Proof:

Show transitivity, subterm property, stability under substitutions, compatibility with Σ-operations, and irreflexivity, usually by induction on the sum of the term sizes and case analysis.

Details: Baader and Nipkow, page 119/120.

Recursive Path Orderings

Theorem:

If the precedence $>$ is total, then the lexicographic path ordering $>_{\mathsf{lpo}}$ is total on ground terms, i. e., for all $s, t \in \mathsf{T}_\Sigma(\emptyset)$: $s >_{\mathsf{lpo}} t \vee t >_{\mathsf{lpo}} s \vee s = t$.

Proof:

By induction on $|s| + |t|$ and case analysis.