The Well-Founded Model – A Quick Introduction

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Various Logic Program Semantics

- Assign "meaning" to a program / knowledge base:
  perfect model, stable models, **well-founded model**
- Normal (logic) programs: negation in rule body allowed.

$$win(X) \leftarrow move(X,Y),\ not\ win(Y) \qquad (1)$$

$$move(c,d) \leftarrow \qquad (2)$$

$$move(a,b) \leftarrow \qquad (3)$$

$$move(b,a) \leftarrow \qquad (4)$$

- **The** well-founded model:

True	Undefined	False
$win(c)$	$win(a)$	$win(d)$
	$win(b)$	

- **Two** stable models:

(i)

True	False
$win(c)$	$win(d)$
$win(a)$	$win(b)$

(ii)

True	False
$win(c)$	$win(d)$
$win(b)$	$win(a)$

More About Well-Founded Models

- See [VanGelder/Ross/Schlipf 89, Przymusinski 91]
- Generally accepted for “reasonable” sceptical reasoning
- “well-behaved” :
  - always exists, stratification not required
  - unique model
  - goal-oriented procedure exists
  - quadratic complexity
- *undef* is assigned to atoms which negatively depend on themselves, and for which no independent “well-founded” derivation exists
- XSB-Prolog system (Warren et. al., top-down system)
- SModels (Niemelä et. al., bottom-up system, also for stable model semantics)

"Building in" Information into Programs

- Program P

$q \leftarrow$	$r \leftarrow \mathit{not}\ s$
$p \leftarrow \mathit{not}\ q, s$	$p \leftarrow \mathit{not}\ p$

- Partial interpretation $\mathcal{J}$

True	Undefined	False
q	p, r	s

- Quotient program $\frac{P}{\mathcal{J}}$

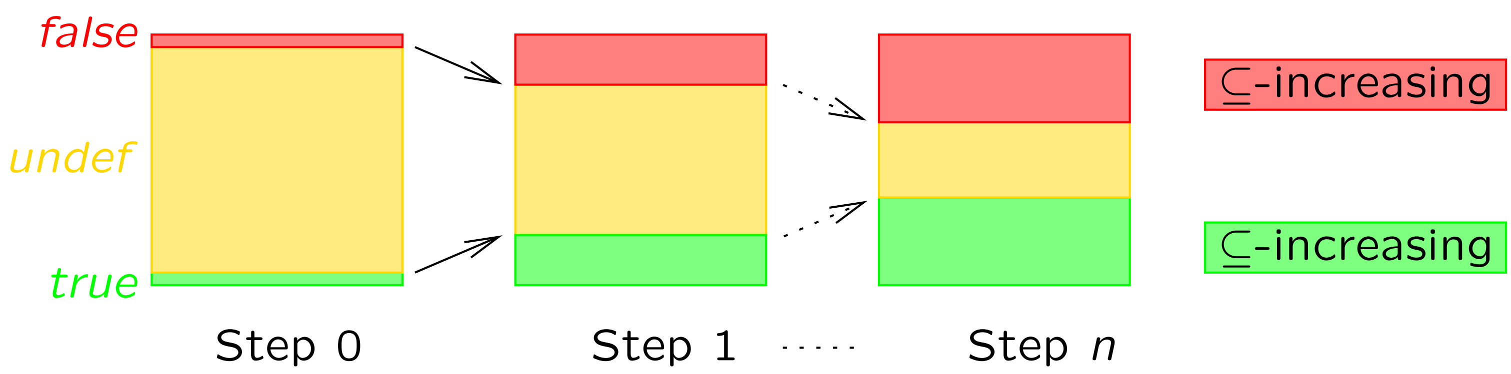
$q \leftarrow$	$r \leftarrow \mathit{true}$
$p \leftarrow \mathit{false}, s$	$p \leftarrow \mathit{undef}$

- $\mathcal{J}$ is a partial model of $\frac{P}{\mathcal{J}}$ iff for all $\mathit{Head} \leftarrow \mathit{Body}$ in $\frac{P}{\mathcal{J}}$:
  - If $\mathcal{J}(\mathit{Body}) = \mathit{true}$ then $\mathcal{J}(\mathit{Head}) = \mathit{true}$
  - If $\mathcal{J}(\mathit{Head}) = \mathit{false}$ then $\mathcal{J}(\mathit{Body}) = \mathit{false}$

- **Least** partial model $LPM(\frac{P}{\mathcal{J}})$
  - $\mathcal{J}$ minimizes *true* atoms, and
  - $\mathcal{J}$ maximizes *false* atoms

True	Undefined	False
q, r	p	s

Well-Founded Models as Fixpoint Iteration



- Maintain two sets to represent $\mathcal{I}_i$:
  - The "*true*" atoms
  - The "*true* or *undef*" atoms
- Set $\mathcal{I}_0 =$ "all *undef*" and do $\mathcal{I}_{i+1} = LPM(\frac{P}{\mathcal{I}_i})$ until fixpoint, where
- seqeuence $(\mathcal{J}_0 =$ "all *false*"$), \mathcal{J}_1, \ldots, \mathcal{J}_{n-1}, (\mathcal{J}_n = \mathcal{J}_{n+1} = LPM(\frac{P}{\mathcal{I}_i}))$ obtained with operator associated to $(Head \leftarrow Body) \in \frac{P}{\mathcal{I}_i}$:

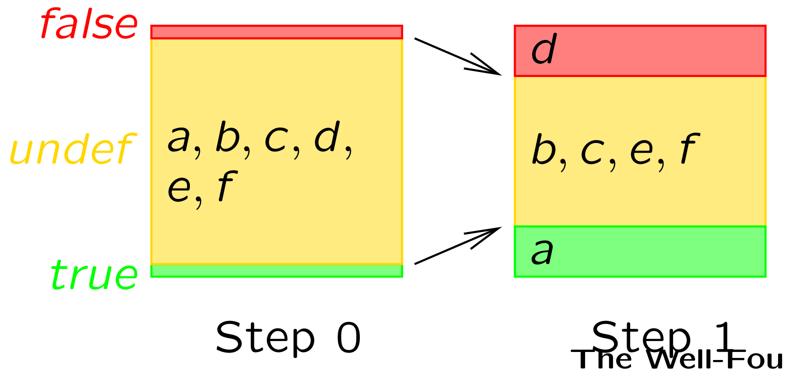
  (i) If $\mathcal{J}_k(Body) = true$ then $\mathcal{J}_{k+1}(Head) = true$

  (ii) If $\mathcal{J}_{k+1}(Head) = false$ then $\mathcal{J}_k(Body) = false$ iff

  If $\underbrace{\mathcal{J}_k(Body) \neq false}_{\mathcal{J}_k(Body) \in \{true, undef\}}$ then $\underbrace{\mathcal{J}_{k+1}(Head) \neq false}_{\mathcal{J}_{k+1}(Head) \in \{true, undef\}}$

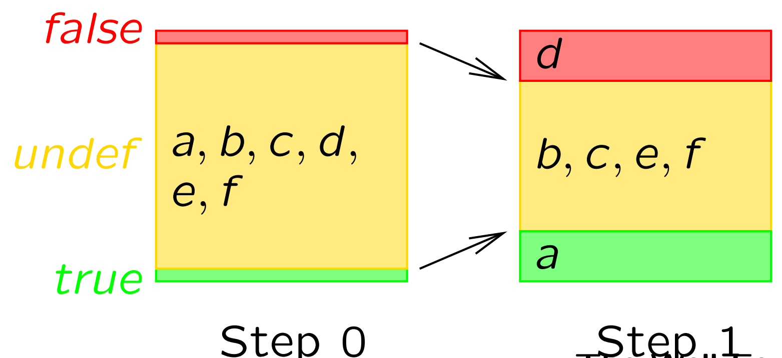
Computing Well-Founded Models, Step 0 $\longmapsto$ Step 1

P
$a \leftarrow$
$c \leftarrow not\ b, a$
$b \leftarrow not\ c$
$e \leftarrow not\ d$
$f \leftarrow e$
$f \leftarrow not\ a$



Computing Well-Founded Models, Step 0 $\mapsto$ Step 1

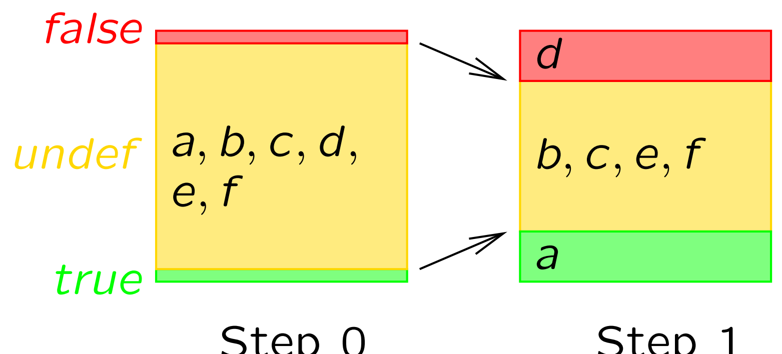
P	(i) build $P/$ a, b, c, d, e, f
$a \leftarrow$	$a \leftarrow$
$c \leftarrow not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow not\ c$	$b \leftarrow undef$
$e \leftarrow not\ d$	$e \leftarrow undef$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow not\ a$	$f \leftarrow undef$



Computing Well-Founded Models, Step 0 $\mapsto$ Step 1

P	(i) build $P/$ [a, b, c, d, e, f]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow \; not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow \; not\ c$	$b \leftarrow undef$
$e \leftarrow \; not\ d$	$e \leftarrow undef$
$f \leftarrow \; e$	$f \leftarrow \; e$
$f \leftarrow \; not\ a$	$f \leftarrow undef$

(ii) derive new *true* atoms [a]

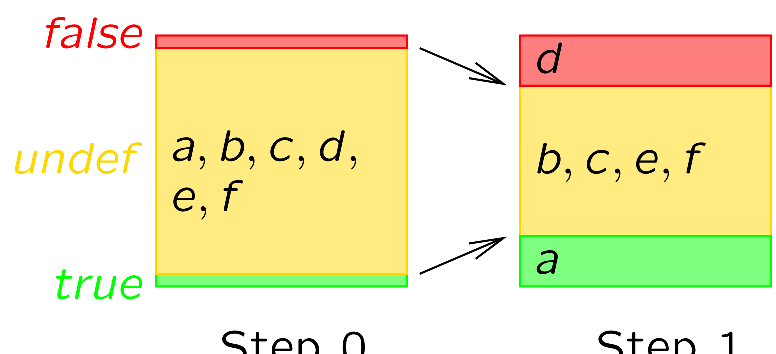


Computing Well-Founded Models, Step 0 $\mapsto$ Step 1

P	(i) build $P/$ [ ][a, b, c, d, e, f][ ]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow not\ c$	$b \leftarrow undef$
$e \leftarrow not\ d$	$e \leftarrow undef$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow not\ a$	$f \leftarrow undef$

(ii) derive new *true* atoms [a]

(iii) derive new *true* or *undef* atoms [a][b, c, e, f]



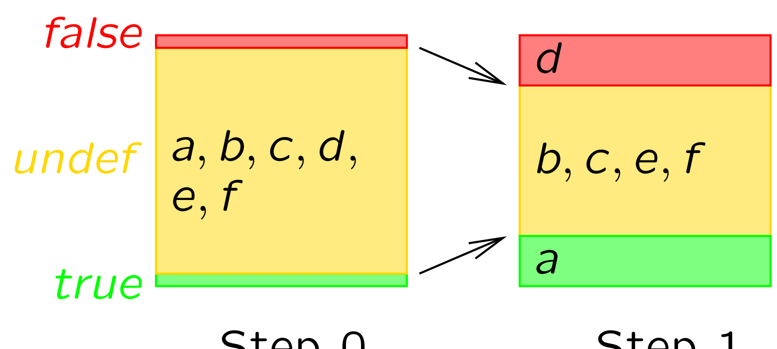
Computing Well-Founded Models, Step 0 $\mapsto$ Step 1

P	(i) build $P/$ [a, b, c, d, e, f]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow \mathit{not}\ b, a$	$c \leftarrow \mathit{undef}, a$
$b \leftarrow \mathit{not}\ c$	$b \leftarrow \mathit{undef}$
$e \leftarrow \mathit{not}\ d$	$e \leftarrow \mathit{undef}$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow \mathit{not}\ a$	$f \leftarrow \mathit{undef}$

(ii) derive new *true* atoms [a]

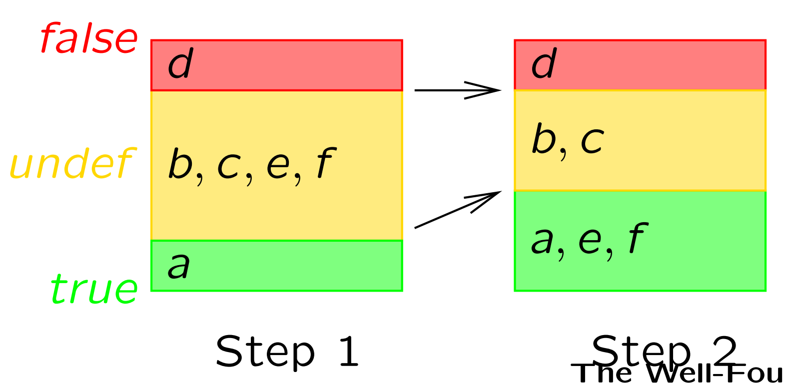
(iii) derive new *true* or *undef* atoms [a] [b, c, e, f]

(iv) conclude new *false* atoms [d]



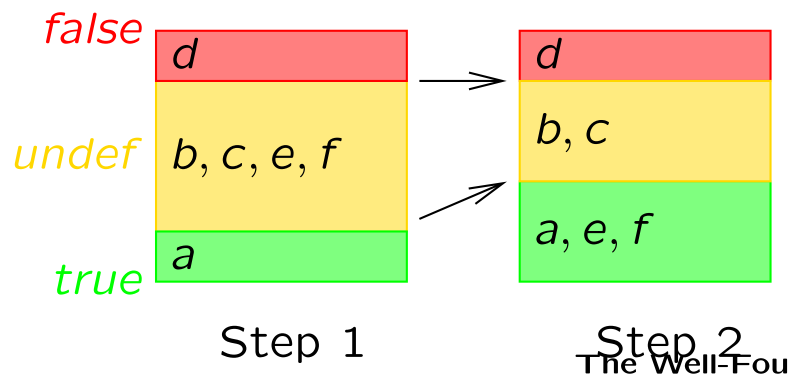
Computing Well-Founded Models, Step 1 $\longmapsto$ Step 2

P
$a \leftarrow$
$c \leftarrow not\ b, a$
$b \leftarrow not\ c$
$e \leftarrow not\ d$
$f \leftarrow e$
$f \leftarrow not\ a$



Computing Well-Founded Models, Step 1 $\mapsto$ Step 2

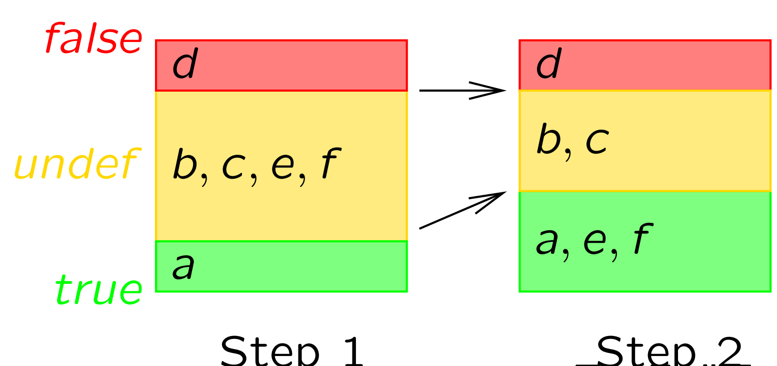
P	(i) build $P/$ a | b, c, e, f | d
$a \leftarrow$	$a \leftarrow$
$c \leftarrow \mathit{not}\ b, a$	$c \leftarrow \mathit{undef}, a$
$b \leftarrow \mathit{not}\ c$	$b \leftarrow \mathit{undef}$
$e \leftarrow \mathit{not}\ d$	$e \leftarrow \mathit{true}$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow \mathit{not}\ a$	$f \leftarrow \mathit{false}$



Computing Well-Founded Models, Step 1 $\mapsto$ Step 2

P	(i) build $P/$ [a] [b, c, e, f] [d]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow not\ c$	$b \leftarrow undef$
$e \leftarrow not\ d$	$e \leftarrow true$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow not\ a$	$f \leftarrow false$

(ii) derive new *true* atoms [a, e, f]

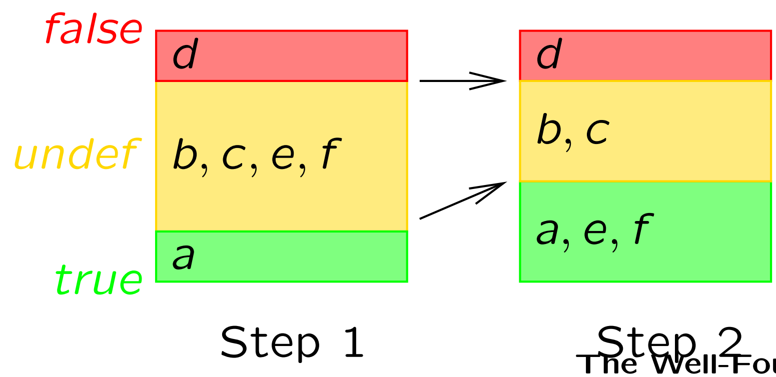


Computing Well-Founded Models, Step 1 $\longmapsto$ Step 2

P	(i) build $P/$ [a | b, c, e, f | d]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow not\ b, a$	$c \leftarrow undef, a$
$b \leftarrow not\ c$	$b \leftarrow undef$
$e \leftarrow not\ d$	$e \leftarrow true$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow not\ a$	$f \leftarrow false$

(ii) derive new *true* atoms [a, e, f]

(iii) derive new *true* or *undef* atoms [a, e, f] [b, c]



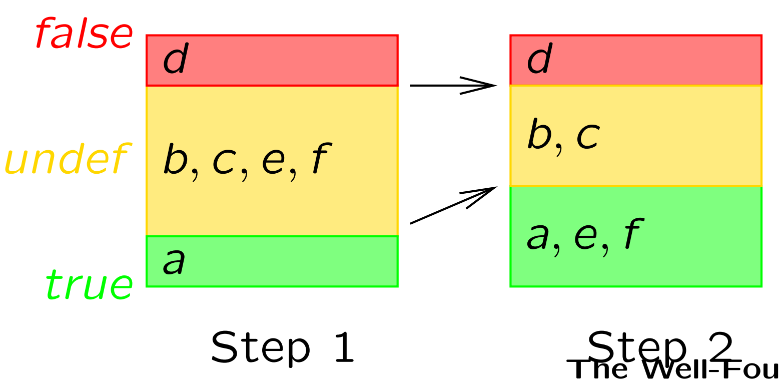
Computing Well-Founded Models, Step 1 $\mapsto$ Step 2

P	(i) build $P/$ [a | b, c, e, f | d]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow \textit{not}\ b, a$	$c \leftarrow \textit{undef}, a$
$b \leftarrow \textit{not}\ c$	$b \leftarrow \textit{undef}$
$e \leftarrow \textit{not}\ d$	$e \leftarrow \textit{true}$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow \textit{not}\ a$	$f \leftarrow \textit{false}$

(ii) derive new *true* atoms [a, e, f]

(iii) derive new *true* or *undef* atoms [a, e, f] [b, c]

(iv) conclude new *false* atoms [d]



Computing Well-Founded Models, Step 1 $\mapsto$ Step 2

P	(i) build $P/$ [a | b, c, e, f | d]
$a \leftarrow$	$a \leftarrow$
$c \leftarrow \textit{not } b, a$	$c \leftarrow \textit{undef}, a$
$b \leftarrow \textit{not } c$	$b \leftarrow \textit{undef}$
$e \leftarrow \textit{not } d$	$e \leftarrow \textit{true}$
$f \leftarrow e$	$f \leftarrow e$
$f \leftarrow \textit{not } a$	$f \leftarrow \textit{false}$

(ii) derive new *true* atoms [a, e, f]

(iii) derive new *true* or *undef* atoms [a, e, f | b, c]

(iv) conclude new *false* atoms [d]

Fixpoint reached - stop