ADFOCS 2017

multilinear world

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introduction

multilinear polynomials

determinant

$$det_n(X) = \sum_{\pi \in S_n} sign(\pi) \prod_{i \in [n]} X_{i,\pi(i)}$$

permanent

$$perm_n(X) = \sum_{\pi \in S_n} \prod_{i \in [n]} X_{i,\pi(i)}$$

symmetric polynomials

$$S_{n,d}(X) = \sum_{T \subseteq [n] : |T| = d} \prod_{i \in T} x_i$$

multilinear complexity [Nisan-Wigderson]

a polynomial is multilinear if individual degrees are at most 1

multilinear circuit[1]

$$v = v_1 \times v_2 \;\Rightarrow\; var(v_1) \cap var(v_2) = \emptyset$$

multilinear ABP: no variable appears twice on $a \to b$ paths

a monotone device for multilinear polynomial is multilinear

what are multilinear complexities of multilinear polynomials?

[1] syntactic

multilinear world

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[Raz, Raz-Y]

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[Raz-Y]

circuits of depth* $d + 1$ are super-poly stronger than depth d

lower bounds

[Raz]

multilinear formulas for det_n or $perm_n$ are of size $n^{\Omega(\log n)}$

[Shpilka-Raz-Y]

$\tilde{\Omega}(n^{4/3})$ multilinear circuit-size lower bound

[Raz-Y]

depth d multilinear circuits for det_n or $perm_n$ are of size $2^{n^{\Omega(1/d)}}$

lower bounds

outline

I. identify a weakness of multilinear formulas

II. exploit it, preferably combinatorially

wish: avoid algebra and argue combinatorially

weakness different grading

lemma

if f is n-variate multilinear formula-size s then

$$f = \sum_{i=1}^{s} g_i$$

where each g_i is log-product:

$$g_i = g_{i,1} g_{i,2} \cdots g_{i,t}$$

with $t = \Omega(\log n)$ and there is a partition of X to $X_{i,1}, \ldots, X_{i,t}$ so that

$$|X_{i,j}| \geq n^{1/2}$$

and

$$var(g_{i,j}) \subseteq X_{i,j}$$

exploiting weakness

to exploit weakness find a "measure" that is

- small on log-product
- sub-additive
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[Nisan] the partial derivative matrix

[Raz] random partitions

partial derivative matrix

given $f \in \mathbb{F}[Y, Z]$ define a matrix $M = M_f$ by

$$M_{p,q} = \text{coefficient of } pq \text{ in } f$$

where p, q are monomials in Y, Z

partitions

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given $\pi : X \to Y \cup Z$ the polynomial

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there are many such matrices for f

can choose one **after** seeing the alleged formula

criterion

f has **full-rank** if for every partition π of X to two parts of equal size the partial derivative matrix M_π has full rank

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theorem [Raz]

if f has full-rank then every multilinear formula for f has size at least $n^{\Omega(\log n)}$

properties of rank

let $f \in \mathbb{F}[Y, Z]$ be multilinear with $|Y| + |Z| = n$

1. if $f = gh$ then $M_f = M_g \otimes M_h$ and
$$rank(M_f) = rank(M_g) rank(M_h)$$
2. if $f = g + h$ then $M_f = M_g + M_h$ and
$$rank(M_f) \leq rank(M_g) + rank(M_h)$$
3. $$rank(M_f) \leq 2^{\min\{|Y|,|Z|\}} \leq 2^{(n-\Delta)/2}$$

where
$$\Delta = \big||Y| - |Z|\big|$$

lemma (random partitions)

X is partitioned to $X_1, \ldots, X_t$ each of size $n_j \geq n^{1/2}$

choose uniformly at random a bijection

$$\pi : X \to Y \cup Z$$

where $|X| = n$ and $|Y| = |Z| = n/2$

then

$$\Pr\left[\Delta_j < n^{1/100} \text{ for all j}\right] < n^{-t/1000}$$

where

$$\Delta_j = \big||Y_j| - |Z_j|\big|$$

and Y_j, Z_j come from $\pi(X_j)$

intuition

X is partitioned to $X_1, \ldots, X_t$ each of size $n_j \geq n^{1/2}$
$\pi : X \to Y \cup Z$ is random
$\Delta_j = \big||Y_j| - |Z_j|\big|$

idea

1. "independence"

$$\Pr\left[\Delta_j < n^{1/100} \text{ for all j}\right] \approx \prod_j \Pr\left[\Delta_j < n^{1/100}\right]$$

2. "anti-concentration"

$$\Pr\left[\Delta_j < n^{1/100}\right] \lesssim \frac{2n^{1/100}}{n_j^{1/2}} \leq n^{-1/1000}$$

LB: the calculation

write $f = \sum_{i=1}^{s} g_i$ where g_i is log-product with $s < n^{\log(n)/1000}$

choose a partition π at random and set $M = M_\pi$

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$$1 = \Pr\left[rank(M) = 2^{n/2}\right] = \Pr\left[rank\left(\sum_i M_i\right) = 2^{n/2}\right]$$

$$\leq \Pr\left[\sum_i rank(M_i) \geq 2^{n/2}\right] \leq \Pr\left[\exists i \ rank(M_i) \geq 2^{n/2-\log s}\right]$$

$$\leq \sum_i \Pr\left[rank(M_i) \geq 2^{n/2-\log s}\right]$$

$$= \sum_i \Pr\left[\prod_j rank(M_{i,j}) \geq 2^{n/2-\log s}\right]$$

$$\leq \sum_i \Pr\left[|\Delta_{i,j}| < n^{1/100} \text{ for all } j\right] \leq s \cdot n^{-\Omega(\log n)}$$

again

if f has multilinear formula of size s

weakness: write f as a sum of s log-product polynomials

randomness: if s is small then there is a partition that makes all log-products of "low rank"

full rank: f has full-rank so s is large full-rank polynomials

[Raz]

both det_n and $perm_n$ are full-rank with respect to some "rich enough" family of partitions

separating circuits and formulas [Raz-Y]

let $X = \{x_1, \ldots, x_n\}$ and Y be extra variables

for $b - a = 1$ define

$$p_{a,b} = x_a + x_b$$

and for $b - a$ odd inductively define

$$p_{a,b} = y_{a,b}(x_a + x_b)p_{a+1,b-1} + \sum_{k:k-a \ odd} y_{a,b,k}p_{a,k}p_{k+1,b}$$

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properties

1. $p_{a,b}$ has a multilinear circuit of size $poly(n)$
2. $p_{a,b}$ is full-rank with respect to $X_{a,b}$ over $\mathbb{F}(Y)$

separating ABPs and formulas [Dvir-Malod-Perifel-Y]

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$$\begin{aligned} p_{a,b} = {} & y_{1,a,b} p_{a+1,b-1} (x_a + x_b) \\ & + y_{2,a,b} p_{a+2,b} (x_a + x_{a+1}) \\ & + y_{3,a,b} p_{a,b-2} (x_{b-1} + x_b) \end{aligned}$$

where addition is modulo n

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properties

1. $p_{a,b}$ has a multilinear ABP of size $poly(n)$
2. $p_{a,b}$ is **not** full-rank with respect to $X_{a,b}$ over $\mathbb{F}(Y)$
3. formula lower bound can still be proved

lower bounds for ABPs?

no strong lower bound for multilinear ABPs

conjecture

if f has full-rank then any multilinear ABP for f has super-poly size summary

many natural multilinear polynomials

multilinear devices are a natural way to compute them

know how to prove strong lower bounds for multilinear formulas

grading multilinear polynomials by number of variables

what about ABPs or circuits?