Complexity of Matrix Multiplication and Bilinear Problems

François Le Gall

Graduate School of Informatics
Kyoto University

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The tensor of matrix multiplication

Definition

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m,n,p\rangle = \sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n} a_{ik}\otimes b_{kj}\otimes c_{ij}.$$

Rank (slightly informal definition):

$R(\langle m,n,p\rangle) =$ minimal t such that $\langle m,n,p\rangle$ can be written as the sum of t terms of the form (lin. comb. of a_{ij}) $\otimes$ (lin. comb. of b_{ij}) $\otimes$ (lin. comb. of c_{ij}).

Theorem

$R(\langle p,p,p\rangle) \leq t \Rightarrow \omega \leq \log_p(t)$

Strassen showed that

$R(\langle 2,2,2\rangle) \leq 7$

$\Rightarrow \omega < 2.81$

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$$R(\langle p, p, p \rangle) \leq t \Rightarrow \omega \leq \log_p(t)$$

typically stated as

Theorem

$$R(\langle p, p, p \rangle) \leq t \Rightarrow p^{\omega} \leq t$$

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harder generalization

Theorem

$\underline{R}(\langle m,n,p\rangle) \leq t \Rightarrow (mnp)^{\omega/3} \leq t$

border rank

$\underline{R}(\langle m,n,p\rangle) \leq R(\langle m,n,p\rangle)$

can be strictly smaller than the rank (and then gives better bounds on ω)

easy generalization

Theorem

$R(\langle m,n,p\rangle) \leq t \Rightarrow (mnp)^{\omega/3} \leq t$

typically stated as

Theorem

$R(\langle p,p,p\rangle) \leq t \Rightarrow p^{\omega} \leq t$

Bini et al. showed that

$$\underline{R}(\langle 3,2,2\rangle) \leq 10$$

$$\Rightarrow \omega < 2.78$$

The asymptotic sum inequality

Schönhage's construction (1981):

$$T_{\text{Schon}} = \underbrace{\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij}}_{\langle 3,1,3 \rangle} + \underbrace{\sum_{k=1}^{4} v_k \otimes v_k \otimes w}_{\langle 1,4,1 \rangle}$$

$\underline{R}(T_{\text{Schon}}) \leq 10$

the sum is direct (the two terms do not share variables)

Theorem (the asymptotic sum inequality, special case) [Schönhage 1981]

$$\underline{R}(\langle m_1, n_1, p_1 \rangle \oplus \langle m_2, n_2, p_2 \rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

Application: $9^{\omega/3} + 4^{\omega/3} \leq 10 \Rightarrow \omega \leq 2.59 \ldots$

Using a variant of this construction, Schönhage finally obtained $\omega \leq 2.54 \ldots$

The asymptotic sum inequality

Theorem (the asymptotic sum inequality, general form) [Schönhage 1981]

$$\underline{R}\left(\bigoplus_{i=1}^{k}\langle m_i, n_i, p_i\rangle\right) \leq t \Longrightarrow \sum_{i=1}^{k}(m_i n_i p_i)^{\omega/3} \leq t$$

↑

Theorem (the asymptotic sum inequality, special case) [Schönhage 1981]

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Application: $9^{\omega/3} + 4^{\omega/3} \leq 10 \Rightarrow \omega \leq 2.59$ ...

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History of the main improvements on the exponent of square matrix multiplication

Upper bound	Year	Authors	
$\omega \leq 3$			
$\omega < 2.81$	1969	Strassen	Rank of a tensor
$\omega < 2.79$	1979	Pan	
$\omega < 2.78$	1979	Bini, Capovani, Romani and Lotti	Border rank of a tensor
$\omega < 2.55$	1981	Schönhage	Asymptotic sum inequality
$\omega < 2.53$	1981	Pan	
$\omega < 2.52$	1982	Romani	
$\omega < 2.50$	1982	Coppersmith and Winograd	
$\omega < 2.48$	1986	Strassen	Laser method
$\omega < 2.376$	1987	Coppersmith and Winograd	
$\omega < 2.374$	2010	Stothers	
$\omega < 2.3729$	2012	Vassilevska Williams	
$\omega < 2.3728639$	2014	LG	

Overview of the Lectures

✓ Fundamental techniques for fast matrix multiplication (1969~1987)

- Basics of bilinear complexity theory: exponent of matrix multiplication, Strassen's algorithm, bilinear algorithms
- First technique: tensor rank and recursion
- Second technique: border rank
- Third technique: the asymptotic sum inequality
- Fourth technique: the laser method

Lecture 1

✓ Recent progress on matrix multiplication (1987~)

- Laser method on powers of tensors
- Other approaches
- Lower bounds
- Rectangular matrix multiplication

Lecture 2

✓ Applications of matrix multiplications, open problems

Lecture 3

The “laser method”

Why this is called the “laser method”?

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limited by our ignorance about ω. Surprisingly, the exact knowledge of the left end of Δ_c can be used to obtain an improved estimate for its right end, namely $\omega < 2.48$. The method employed is called *laser method* [27], since it is reminiscent of the generation of coherent light.

from V. Strassen.
Algebra and Complexity.
Proceedings of the first European Congress of Mathematics, pp. 429-446, 1994.

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	$\omega < 2.373$	2010	Stothers
	$\omega < 2.3729$	2012	Vassilevska Williams
	$\omega < 2.3728639$	2014	Le Gall

The first CW construction (Section 5.1)

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Coppersmith and Winograd (1987) introduced the following tensor:

$$T_{\text{easy}} = T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110},$$

where

$$T_{\text{easy}}^{011} = \sum_{i=1}^{q} x_0 \otimes y_i \otimes z_i$$

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$$T_{\text{easy}}^{101} = \sum_{i=1}^{q} x_i \otimes y_0 \otimes z_i \cong \langle q, 1, 1 \rangle \qquad T_{\text{easy}}^{101} = \sum_{i=1}^{q} x_{i0} \otimes y_{00} \otimes z_{i0}$$

$$T_{\text{easy}}^{110} = \sum_{i=1}^{q} x_i \otimes y_i \otimes z_0 \cong \langle 1, q, 1 \rangle \qquad T_{\text{easy}}^{110} = \sum_{i=1}^{q} x_{0i} \otimes y_{i0} \otimes z_{00}$$

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1×1 matrix by 1×q matrix

$$T_{\text{easy}}^{101} = \sum_{i=1}^{q} x_i \otimes y_0 \otimes z_i \cong \langle q, 1, 1 \rangle \qquad T_{\text{easy}}^{101} = \sum_{i=1}^{q} x_{i0} \otimes y_{00} \otimes z_{i0}$$

q×1 matrix by 1×1 matrix

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1×q matrix by q×1 matrix

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$$V = V_0 \oplus V_1, \quad \text{where } V_0 = span\{y_0\} \text{ and } V_1 = span\{y_1, \ldots, y_q\}$$
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tensor over (U, V, W)

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Coppersmith and Winograd (1987) introduced the following tensor:

$$T_{\mathsf{easy}} = T^{011}_{\mathsf{easy}} + T^{101}_{\mathsf{easy}} + T^{110}_{\mathsf{easy}},$$

← This is not a direct sum

where

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Coppersmith and Winograd showed that

$$\underline{R}(T_{\text{easy}}) \le q + 2$$

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$$\lambda^3 T_{\text{easy}} = T' + \lambda^4 T''$$

where $T' = \sum_{i=1}^{q} \lambda(x_0 + \lambda x_i) \otimes (y_0 + \lambda y_i) \otimes (z_0 + \lambda z_i)$

$$- (x_0 + \lambda^2 \sum_{i=1}^{q} x_i) \otimes (y_0 + \lambda^2 \sum_{i=1}^{q} y_i) \otimes (z_0 + \lambda^2 \sum_{i=1}^{q} z_i)$$

$$+ (1 - q\lambda) x_0 \otimes y_0 \otimes z_0$$

$q + 2$ multiplications

and T'' is some tensor

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Consider $T^{\otimes 2}_{\text{easy}}$

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Consider $T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$

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$= T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} + T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}$ (9 terms)

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Since the sum in not direct, we cannot use the asymptotic sum inequality

Consider $T^{\otimes 2}_{\text{easy}} = (T^{011}_{\text{easy}} + T^{101}_{\text{easy}} + T^{110}_{\text{easy}}) \otimes (T^{011}_{\text{easy}} + T^{101}_{\text{easy}} + T^{110}_{\text{easy}})$

$$= T^{011}_{\text{easy}} \otimes T^{011}_{\text{easy}} + T^{011}_{\text{easy}} \otimes T^{101}_{\text{easy}} + \cdots + T^{110}_{\text{easy}} \otimes T^{110}_{\text{easy}}$$ (9 terms)

Consider $T^{\otimes N}_{\text{easy}} = T^{011}_{\text{easy}} \otimes \cdots \otimes T^{011}_{\text{easy}} + \cdots + T^{110}_{\text{easy}} \otimes \cdots \otimes T^{110}_{\text{easy}}$ (3^N terms)

The first CW construction (Section 5.1)

$$T_{\text{easy}} = T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}$$

$$T_{\text{easy}}^{011} = \sum_{i=1}^{q} x_0 \otimes y_i \otimes z_i \cong \langle 1, 1, q \rangle$$

$$T_{\text{easy}}^{101} = \sum_{i=1}^{q} x_i \otimes y_0 \otimes z_i \cong \langle q, 1, 1 \rangle$$

$$T_{\text{easy}}^{110} = \sum_{i=1}^{q} x_i \otimes y_i \otimes z_0 \cong \langle 1, q, 1 \rangle$$

Coppersmith and Winograd showed that

$$\underline{R}(T_{\text{easy}}) \leq q + 2$$

Since the sum in not direct, we cannot use the asymptotic sum inequality

Consider $T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$

$= T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} + T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}$ (9 terms)

Consider $T_{\text{easy}}^{\otimes N} = T_{\text{easy}}^{011} \otimes \cdots \otimes T_{\text{easy}}^{011} + \cdots + T_{\text{easy}}^{110} \otimes \cdots \otimes T_{\text{easy}}^{110}$ (3^N terms)

Coppersmith and Winograd showed how to select $\approx \left(\frac{3}{2^{2/3}}\right)^N$ terms that do not share variables (i.e., form a direct sum)

The first CW construction (Section 5.1)

Theorem 4

The tensor $T_{\text{easy}}^{\otimes N}$ can be converted into a direct sum of

$$2^{\left(H\left(\frac{1}{3},\frac{2}{3}\right)-o(1)\right)N}$$

terms, each containing $\frac{N}{3}$ copies of T_{easy}^{011}, $\frac{N}{3}$ copies of T_{easy}^{101} and $\frac{N}{3}$ copies of T_{easy}^{110}.

The first CW construction (Section 5.1)

Theorem 4

The tensor $T_{\text{easy}}^{\otimes N}$ can be converted into a direct sum of

$$2^{\left(H\left(\frac{1}{3},\frac{2}{3}\right)-o(1)\right)N} = \left(\frac{3}{2^{2/3}}\right)^{(1-o(1))N}$$

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The first CW construction (Section 5.1)

$H\left(\frac{1}{3},\frac{2}{3}\right) = -\frac{1}{3}\log_2\left(\frac{1}{3}\right) - \frac{2}{3}\log_2\left(\frac{2}{3}\right)$ (the entropy)

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The first CW construction (Section 5.1)

$$H\left(\tfrac{1}{3},\tfrac{2}{3}\right) = -\tfrac{1}{3}\log_2\left(\tfrac{1}{3}\right) - \tfrac{2}{3}\log_2\left(\tfrac{2}{3}\right) \quad \text{(the entropy)}$$
$$= \log_2\left(3^{1/3}\times\left(\tfrac{3}{2}\right)^{2/3}\right) = \log_2\left(\tfrac{3}{2^{2/3}}\right)$$

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Consider $T_{\text{easy}}^{\otimes N} = \underbrace{T_{\text{easy}}^{011} \otimes \cdots \otimes T_{\text{easy}}^{011}} + \cdots + \underbrace{T_{\text{easy}}^{110} \otimes \cdots \otimes T_{\text{easy}}^{110}}$ (3^N terms)

N copies of T_{easy}^{011}
0 copies of T_{easy}^{101}
0 copies of T_{easy}^{110}

0 copies of T_{easy}^{011}
0 copies of T_{easy}^{101}
N copies of T_{easy}^{110}

The first CW construction (Section 5.1)

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Remember:

$$T_{\text{easy}}^{011} = \sum_{i=1}^{q} x_0 \otimes y_i \otimes z_i \cong \langle 1, 1, q\rangle$$

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Theorem 3 (the asymptotic sum inequality, general form) [Schönhage 1981]

$$\underline{R}\left(\bigoplus_{i=1}^{k} \langle m_i, n_i, p_i \rangle\right) \leq t \Longrightarrow \sum_{i=1}^{k} (m_i n_i p_i)^{\omega/3} \leq t$$

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Consequence: $\left(\frac{3}{2^{2/3}}\right)^{(1-o(1))N} \times q^{N\omega/3} \leq \underline{R}(T_{\text{easy}}^{\otimes N})$

The first CW construction (Section 5.1)

Theorem 4

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Consequence: $\left(\frac{3}{2^{2/3}}\right)^{(1-o(1))N} \times q^{N\omega/3} \leq \underline{R}(T_{\text{easy}}^{\otimes N}) \leq \underline{R}(T_{\text{easy}})^N = (q+2)^N$

The first CW construction (Section 5.1)

Theorem 4

The tensor $T_{\text{easy}}^{\otimes N}$ can be converted into a direct sum of

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The first CW construction (Section 5.1)

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$$\Longrightarrow \frac{3}{2^{2/3}} \times q^{\omega/3} \le q+2 \Longrightarrow \omega \le 2.403\ldots \text{ for } q = 8$$

Idea behind the proof of Theorem 4

$$T_{\text{easy}} = T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}$$

$$T_{\text{easy}}^{011} = \sum_{i=1}^{q} x_0 \otimes y_i \otimes z_i$$

$$T_{\text{easy}}^{101} = \sum_{i=1}^{q} x_i \otimes y_0 \otimes z_i$$

$$T_{\text{easy}}^{110} = \sum_{i=1}^{q} x_i \otimes y_i \otimes z_0$$

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} + T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110} \quad \text{(9 terms)}$$

Idea behind the proof of Theorem 4

$$T_{\text{easy}} = T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}$$

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$$= T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110} \quad \text{(9 terms)}$$

$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'})$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

$$T_{\text{easy}} = T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}$$

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$$T_{\text{easy}}^{110} = \sum_{i=1}^{q} x_i \otimes y_i \otimes z_0$$

Consider $N = 2$

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

$$T_{\text{easy}} = T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}$$

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$$= T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}$$ (9 terms)

011011

$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'})$$ ⟶ label 011011

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

$$T_{\text{easy}} = T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}$$

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Consider $N = 2$

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$$= \underbrace{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}}_{001111} + \boxed{\underbrace{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}}_{011011}} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}$$ (9 terms)

$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

$$T_{\text{easy}} = T^{011}_{\text{easy}} + T^{101}_{\text{easy}} + T^{110}_{\text{easy}}$$

$$T^{011}_{\text{easy}} = \sum_{i=1}^{q} x_0 \otimes y_i \otimes z_i$$

$$T^{101}_{\text{easy}} = \sum_{i=1}^{q} x_i \otimes y_0 \otimes z_i$$

$$T^{110}_{\text{easy}} = \sum_{i=1}^{q} x_i \otimes y_i \otimes z_0$$

Consider $N = 2$

$$T^{\otimes 2}_{\text{easy}} = (T^{011}_{\text{easy}} + T^{101}_{\text{easy}} + T^{110}_{\text{easy}}) \otimes (T^{011}_{\text{easy}} + T^{101}_{\text{easy}} + T^{110}_{\text{easy}})$$

$$= \underbrace{T^{011}_{\text{easy}} \otimes T^{011}_{\text{easy}}}_{001111} + \boxed{\underbrace{T^{011}_{\text{easy}} \otimes T^{101}_{\text{easy}}}_{011011}} + \cdots + \underbrace{T^{110}_{\text{easy}} \otimes T^{110}_{\text{easy}}}_{111100}$$ (9 terms)

$$T^{011}_{\text{easy}} \otimes T^{101}_{\text{easy}} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'})$$ ⟶ label 011011

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

$$T_{\text{easy}} = T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}$$

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$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}$$ (9 terms)

001111 011011 111100

011110 100111 110011

110110 101101 111001

$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'})$$ ⟶ label 011011

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Idea behind the proof of Theorem 4

$$T_{\text{easy}} = T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}$$

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$$T_{\text{easy}}^{110} = \sum_{i=1}^{q} x_i \otimes y_i \otimes z_0$$

We cannot apply the asymptotic sum inequality since the sum is not direct

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}$$ (9 terms)

001111 011011 111100
011110 100111 110011
110110 101101 111001

$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} + T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110} \quad \text{(9 terms)}$$

001111 011011 111100

110110 011110 101101 100111 111001 110011

$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

$$T^{011}_{\text{easy}} \otimes T^{011}_{\text{easy}} = \sum_{i,i'=0}^{q} (x_0 \otimes x_0) \otimes (y_i \otimes y_{i'}) \otimes (z_i \otimes z_{i'})$$

tensor over $(U_0 \otimes U_0) \otimes (V_1 \otimes V_1) \otimes (W_1 \otimes W_1)$

Consider $N = 2$

$$T^{\otimes 2}_{\text{easy}} = (T^{011}_{\text{easy}} + T^{101}_{\text{easy}} + T^{110}_{\text{easy}}) \otimes (T^{011}_{\text{easy}} + T^{101}_{\text{easy}} + T^{110}_{\text{easy}})$$

$$= T^{011}_{\text{easy}} \otimes T^{011}_{\text{easy}} + T^{011}_{\text{easy}} \otimes T^{101}_{\text{easy}} + \cdots + T^{110}_{\text{easy}} \otimes T^{110}_{\text{easy}}$$ (9 terms)

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110110 011110 101101 100111 111001 110011

$$T^{011}_{\text{easy}} \otimes T^{101}_{\text{easy}} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} = \sum_{i,i'=0}^{q} (x_0 \otimes x_0) \otimes (y_i \otimes y_{i'}) \otimes (z_i \otimes z_{i'})$$

tensor over $(U_0 \otimes U_0) \otimes (V_1 \otimes V_1) \otimes (W_1 \otimes W_1)$

SHARE VARIABLES

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} + T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}$$ (9 terms)

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} = \sum_{i,i'=0}^{q} (x_0 \otimes x_0) \otimes (y_i \otimes y_{i'}) \otimes (z_i \otimes z_{i'})$$

tensor over $(U_0 \otimes U_0) \otimes (V_1 \otimes V_1) \otimes (W_1 \otimes W_1)$

SHARE VARIABLES

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011} + T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110} \quad \text{(9 terms)}$$

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110110   011110   101101   100111   111001   110011

$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

$$T^{011}_{\text{easy}} \otimes T^{011}_{\text{easy}} = \sum_{i,i'=0}^{q} (x_0 \otimes x_0) \otimes (y_i \otimes y_{i'}) \otimes (z_i \otimes z_{i'})$$

tensor over $(U_0 \otimes U_0) \otimes (V_1 \otimes V_1) \otimes (W_1 \otimes W_1)$

remove this tensor (by setting $x_0 \otimes x_0$ to zero)

SHARE VARIABLES

Consider $N = 2$

$$T^{\otimes 2}_{\text{easy}} = (T^{011}_{\text{easy}} + T^{101}_{\text{easy}} + T^{110}_{\text{easy}}) \otimes (T^{011}_{\text{easy}} + T^{101}_{\text{easy}} + T^{110}_{\text{easy}})$$

$$= ~~T^{011}_{\text{easy}} \otimes T^{011}_{\text{easy}}~~ + T^{011}_{\text{easy}} \otimes T^{101}_{\text{easy}} + \cdots + T^{110}_{\text{easy}} \otimes T^{110}_{\text{easy}}$$ (9 terms)

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$$T^{011}_{\text{easy}} \otimes T^{101}_{\text{easy}} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \cancel{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110} \quad \text{(9 terms)}$$

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \sout{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}}$$ (9 terms)

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \cancel{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}} \quad \text{(9 terms)}$$

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \cancel{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}}$$ (9 terms)

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \cancel{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}}$$ (9 terms)

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \cancel{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}}$$ (9 terms)

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \cancel{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}} \quad \text{(9 terms)}$$

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \sout{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}}$$ (9 terms)

Labels: ~~001111~~, 011011, 111100, ~~110110~~, ~~011110~~, ~~101101~~, ~~100111~~, ~~111001~~, 110011

$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \cancel{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}}$$ (9 terms)

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Conclusion: we can convert $T_{\text{easy}}^{\otimes 2}$ (a sum of 9 terms) into a direct sum of 2 terms

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \cancel{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}} \quad \text{(9 terms)}$$

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Conclusion: we can convert $T_{\text{easy}}^{\otimes 2}$ (a sum of 9 terms) into a direct sum of 2 terms

we can then apply the asymptotic sum inequality, but this does not give any interesting bound

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \cancel{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}} \quad \text{(9 terms)}$$

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Conclusion: we can convert $T_{\text{easy}}^{\otimes 2}$ (a sum of 9 terms) into a direct sum of 2 terms

we can then apply the asymptotic sum inequality, but this does not give any interesting bound

Consider $N = 2$

$$T_{\text{easy}}^{\otimes 2} = (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110}) \otimes (T_{\text{easy}}^{011} + T_{\text{easy}}^{101} + T_{\text{easy}}^{110})$$

$$= \cancel{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{011}} + \boxed{T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101}} + \cdots + \boxed{T_{\text{easy}}^{110} \otimes T_{\text{easy}}^{110}} \quad \text{(9 terms)}$$

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$$T_{\text{easy}}^{011} \otimes T_{\text{easy}}^{101} = \sum_{i,i'=0}^{q} (x_0 \otimes x_{i'}) \otimes (y_i \otimes y_0) \otimes (z_i \otimes z_{i'}) \longrightarrow \text{label } 011011$$

tensor over $(U_0 \otimes U_1) \otimes (V_1 \otimes V_0) \otimes (W_1 \otimes W_1)$

Idea behind the proof of Theorem 4

Conclusion: we can convert $T_{\text{easy}}^{\otimes 2}$ (a sum of 9 terms) into a direct sum of 2 terms

we can then apply the asymptotic sum inequality, but this does not give any interesting bound

NEXT STEP

Consider $T_{\text{easy}}^{\otimes N} = T_{\text{easy}}^{011} \otimes \cdots \otimes T_{\text{easy}}^{011} + \cdots + T_{\text{easy}}^{110} \otimes \cdots \otimes T_{\text{easy}}^{110}$ (3^N terms)

labels: $\underbrace{0 \cdots 0 1 \cdots 1 1 \cdots 1}_{3N}$ $\qquad$ $\underbrace{1 \cdots 1 1 \cdots 1 0 \cdots 0}_{3N}$

Idea behind the proof of Theorem 4

Theorem 4

The tensor $T_{\text{easy}}^{\otimes N}$ can be converted into a direct sum of

$$2^{\left(H\left(\frac{1}{3},\frac{2}{3}\right)-o(1)\right)N} = \left(\frac{3}{2^{2/3}}\right)^{(1-o(1))N}$$

terms, each containing $\frac{N}{3}$ copies of T_{easy}^{011}, $\frac{N}{3}$ copies of T_{easy}^{101} and $\frac{N}{3}$ copies of T_{easy}^{110}.

Theorem 4 shows that one can keep $\approx \left(\frac{3}{2^{2/3}}\right)^N$ such labels, under the condition that a blue (or red, or green) part cannot be used more than once

Idea behind the proof of Theorem 4

Theorem 4

The tensor $T_{\text{easy}}^{\otimes N}$ can be converted into a direct sum of

$$2^{\left(H\left(\frac{1}{3},\frac{2}{3}\right)-o(1)\right)N} = \left(\frac{3}{2^{2/3}}\right)^{(1-o(1))N}$$

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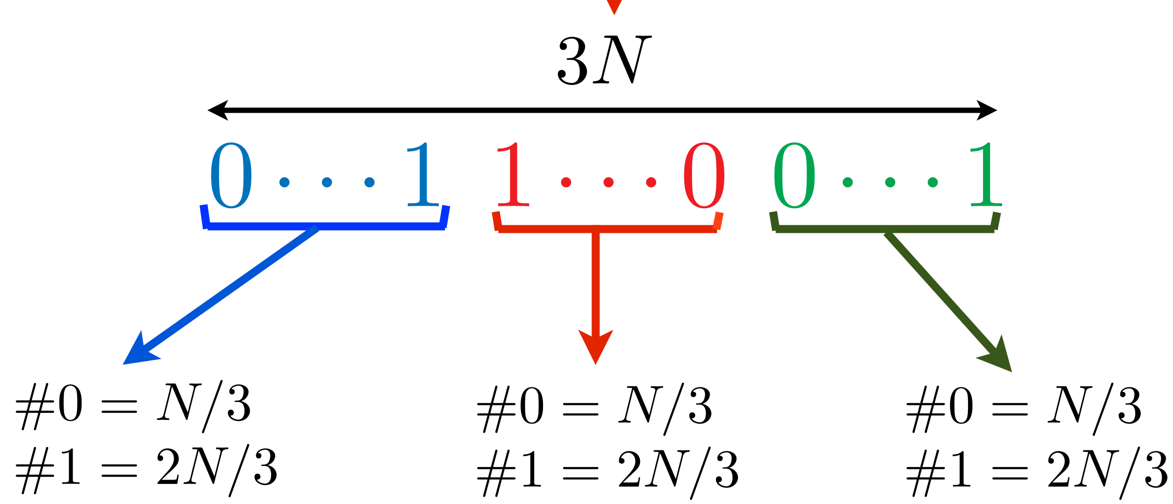

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Idea behind the proof of Theorem 4

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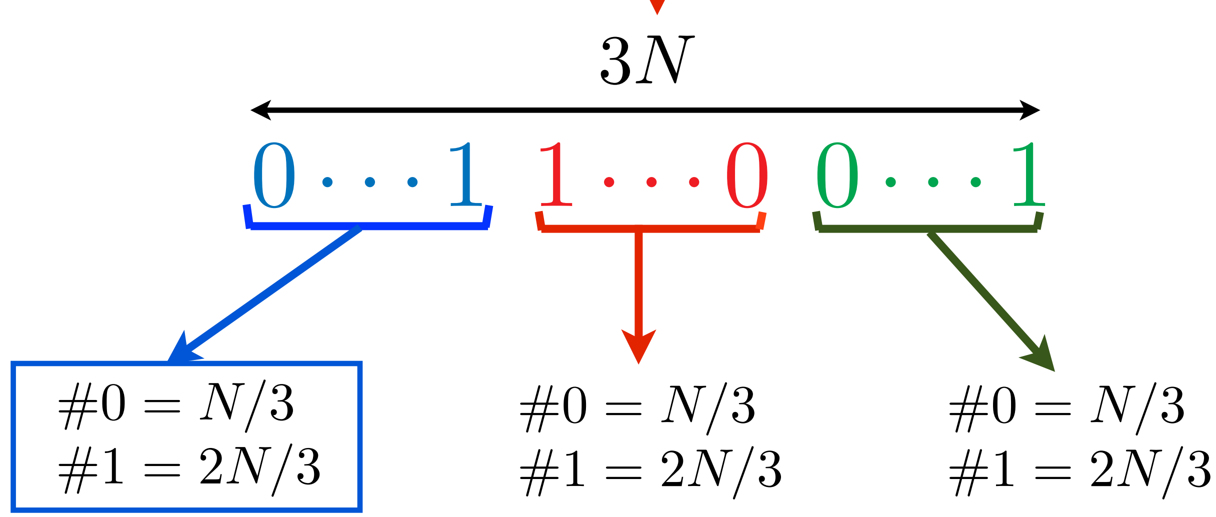

number of possibilities

$$\binom{N}{\frac{N}{3},\frac{2N}{3}} \approx 2^{H\left(\frac{1}{3},\frac{2}{3}\right)N}$$

Theorem 4 shows that one can keep $\approx \left(\frac{3}{2^{2/3}}\right)^{N}$ such labels, under the condition that a blue (or red, or green) part cannot be used more than once

Idea behind the proof of Theorem 4

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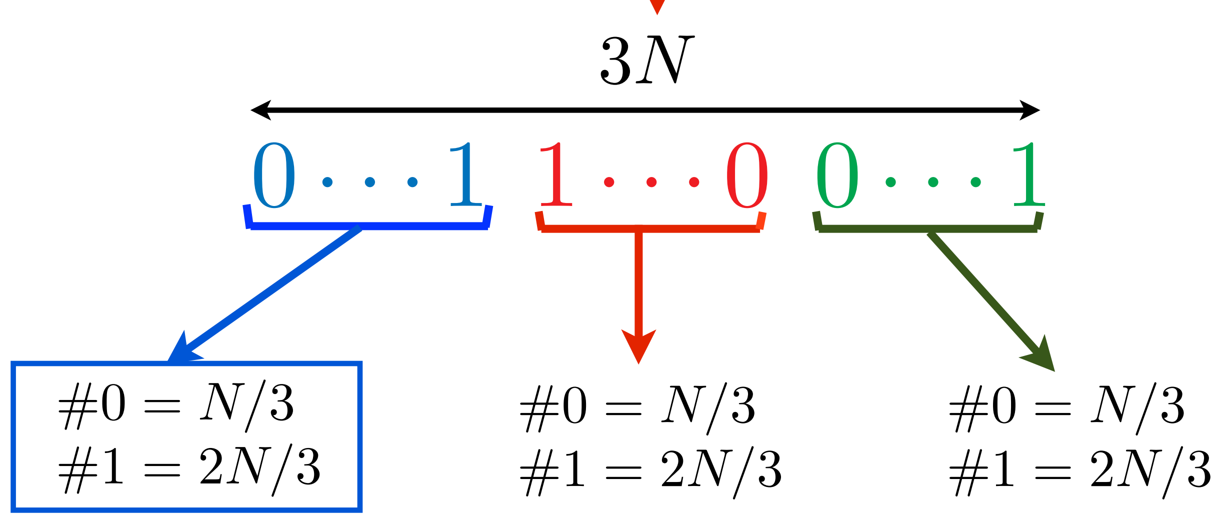

number of possibilities

$$\binom{N}{\frac{N}{3}, \frac{2N}{3}} \approx 2^{H\left(\frac{1}{3},\frac{2}{3}\right)N}$$

Theorem 4 shows that one can keep $\approx \left(\frac{3}{2^{2/3}}\right)^N$ such labels, under the condition that a blue (or red, or green) part cannot be used more than once

The proof of Theorem 4 is based on a complex argument using the existence of dense sets of integers with no three-term arithmetic progression

The second CW construction (Section 5.2)

Let q be a positive integer.

Consider three vector spaces U, V and W of dimension $q+2$ over $\mathbb{F}$.

$U = span\{x_0, \ldots, x_q, x_{q+1}\}$ $W = span\{z_0, \ldots, z_q, z_{q+1}\}$

$V = span\{y_0, \ldots, y_q, y_{q+1}\}$

The second CW construction (Section 5.2)

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$V = span\{y_0, \ldots, y_q, y_{q+1}\}$

Coppersmith and Winograd (1987) considered the following tensor:

$$T_{\text{CW}} = T_{\text{CW}}^{011} + T_{\text{CW}}^{101} + T_{\text{CW}}^{110} + T_{\text{CW}}^{002} + T_{\text{CW}}^{020} + T_{\text{CW}}^{200},$$

where

$$T_{\text{CW}}^{011} = T_{\text{easy}}^{011}$$
$$T_{\text{CW}}^{101} = T_{\text{easy}}^{101}$$
$$T_{\text{CW}}^{110} = T_{\text{easy}}^{110}$$

and

$$T_{\text{CW}}^{002} = x_0 \otimes y_0 \otimes z_{q+1} \cong \langle 1,1,1 \rangle$$
$$T_{\text{CW}}^{020} = x_0 \otimes y_{q+1} \otimes z_0 \cong \langle 1,1,1 \rangle$$
$$T_{\text{CW}}^{200} = x_{q+1} \otimes y_0 \otimes z_0 \cong \langle 1,1,1 \rangle.$$

The second CW construction (Section 5.2)

$$T_{\mathsf{CW}} = T_{\mathsf{easy}} + T_{\mathsf{CW}}^{002} + T_{\mathsf{CW}}^{020} + T_{\mathsf{CW}}^{200}$$

$$T_{\mathsf{CW}}^{002} = x_0 \otimes y_0 \otimes z_{q+1}$$
$$T_{\mathsf{CW}}^{020} = x_0 \otimes y_{q+1} \otimes z_0$$
$$T_{\mathsf{CW}}^{200} = x_{q+1} \otimes y_0 \otimes z_0$$

The second CW construction (Section 5.2)

$$T_{\mathsf{CW}} = T_{\mathsf{easy}} + T_{\mathsf{CW}}^{002} + T_{\mathsf{CW}}^{020} + T_{\mathsf{CW}}^{200}$$

$$T_{\mathsf{CW}}^{002} = x_0 \otimes y_0 \otimes z_{q+1}$$
$$T_{\mathsf{CW}}^{020} = x_0 \otimes y_{q+1} \otimes z_0$$
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$$\underline{R}(T_{\mathsf{CW}}) \leq q + 2$$

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$$\lambda^3 T_{\mathsf{easy}} = T' + \lambda^4 T''$$

where
$$T' = \sum_{i=1}^{q} \lambda(x_0 + \lambda x_i) \otimes (y_0 + \lambda y_i) \otimes (z_0 + \lambda z_i)$$
$$- (x_0 + \lambda^2 \sum_{i=1}^{q} x_i) \otimes (y_0 + \lambda^2 \sum_{i=1}^{q} y_i) \otimes (z_0 + \lambda^2 \sum_{i=1}^{q} z_i)$$
$$+ (1 - q\lambda)(x_0 + \lambda^3 x_{q+1}) \otimes (y_0 + \lambda^3 y_{q+1}) \otimes (z_0 + \lambda^3 z_{q+1})$$

q + 2 multipli-cations

and T'' is some tensor

The second CW construction (Section 5.2)

$$U = span\{x_0, \ldots, x_q, x_{q+1}\} \qquad W = span\{z_0, \ldots, z_q, z_{q+1}\}$$
$$V = span\{y_0, \ldots, y_q, y_{q+1}\}$$

$U = U_0 \oplus U_1 \oplus U_2$, where $U_0 = span\{x_0\}$, $U_1 = span\{x_1, \ldots, x_q\}$ and $U_2 = span\{x_{q+1}\}$

$V = V_0 \oplus V_1 \oplus V_2$, where $V_0 = span\{y_0\}$, $V_1 = span\{y_1, \ldots, y_q\}$ and $V_2 = span\{y_{q+1}\}$

$W = W_0 \oplus W_1 \oplus W_2$, where $W_0 = span\{z_0\}$, $W_1 = span\{z_1, \ldots, z_q\}$ and $W_2 = span\{z_{q+1}\}$

$$T_{\mathsf{CW}} = T_{\mathsf{CW}}^{011} + T_{\mathsf{CW}}^{101} + T_{\mathsf{CW}}^{110} + T_{\mathsf{CW}}^{002} + T_{\mathsf{CW}}^{020} + T_{\mathsf{CW}}^{200}$$

The second CW construction (Section 5.2)

$$U = span\{x_0, \ldots, x_q, x_{q+1}\} \qquad W = span\{z_0, \ldots, z_q, z_{q+1}\}$$
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$$T_{\mathsf{CW}} = T_{\mathsf{CW}}^{011} + T_{\mathsf{CW}}^{101} + T_{\mathsf{CW}}^{110} + T_{\mathsf{CW}}^{002} + T_{\mathsf{CW}}^{020} + T_{\mathsf{CW}}^{200}$$

T_{CW}^{011} tensor over (U_0, V_1, W_1)

T_{CW}^{101} tensor over (U_1, V_0, W_1)

T_{CW}^{110} tensor over (U_1, V_1, W_0)

The second CW construction (Section 5.2)

$$U = span\{x_0, \ldots, x_q, x_{q+1}\} \qquad W = span\{z_0, \ldots, z_q, z_{q+1}\}$$
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$T_{\mathsf{CW}}^{200} = x_{q+1} \otimes y_0 \otimes z_0$ tensor over (U_2, V_0, W_0)

The second CW construction (Section 5.2)

$$U = span\{x_0, \ldots, x_q, x_{q+1}\} \qquad W = span\{z_0, \ldots, z_q, z_{q+1}\}$$
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This is not a direct sum

T_{CW}^{011} tensor over (U_0, V_1, W_1)

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$T_{\mathsf{CW}}^{200} = x_{q+1} \otimes y_0 \otimes z_0$ tensor over (U_2, V_0, W_0)

Analysis of the second construction

$$T_{\mathsf{CW}}^{\otimes N} = (T_{\mathsf{CW}}^{011} + T_{\mathsf{CW}}^{101} + T_{\mathsf{CW}}^{110} + T_{\mathsf{CW}}^{002} + T_{\mathsf{CW}}^{020} + T_{\mathsf{CW}}^{200})^{\otimes N} \quad (6^N \text{ terms})$$

Analysis of the second construction

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Theorem 5

For any $0 \leq \alpha \leq 1/3$ and for N large enough, the tensor $T_{\text{CW}}^{\otimes N}$ can be converted into a direct sum of

$$2^{\left(H\left(\frac{2}{3}-\alpha, 2\alpha, \frac{1}{3}-\alpha\right)-o(1)\right)N}$$

terms, each isomorphic to

$$[T_{\text{CW}}^{011}]^{\otimes \alpha N} \otimes [T_{\text{CW}}^{101}]^{\otimes \alpha N} \otimes [T_{\text{CW}}^{110}]^{\otimes \alpha N} \otimes [T_{\text{CW}}^{002}]^{\otimes \left(\frac{1}{3}-\alpha\right)N} \otimes [T_{\text{CW}}^{020}]^{\otimes \left(\frac{1}{3}-\alpha\right)N} \otimes [T_{\text{CW}}^{200}]^{\otimes \left(\frac{1}{3}-\alpha\right)N}.$$

Analysis of the second construction

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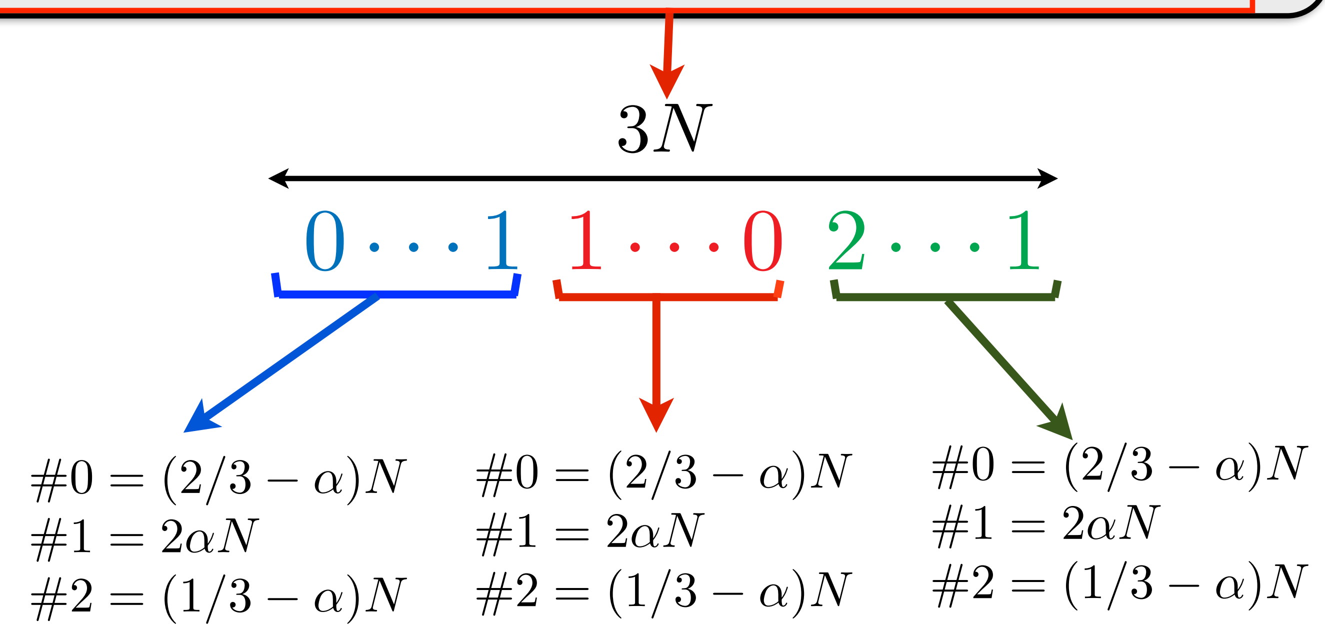

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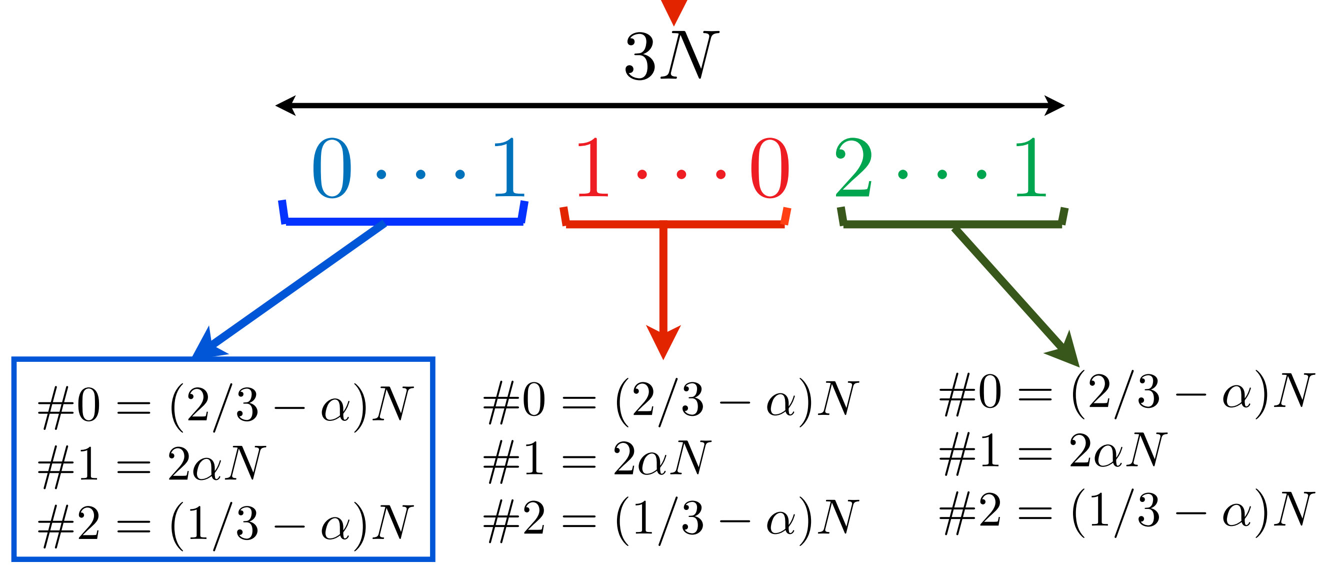

number of possibilities

$$\binom{N}{(\frac{2}{3}-\alpha)N, 2\alpha N, (\frac{1}{3}-\alpha)N} \approx 2^{H\left(\frac{2}{3}-\alpha, 2\alpha, \frac{1}{3}-\alpha\right)N}$$

Analysis of the second construction

Theorem 5

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For any $0 \leq \alpha \leq 1/3$ and for N large enough, the tensor $T_{\mathsf{CW}}^{\otimes N}$ can be converted into a direct sum of

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terms, each isomorphic to

$$\boxed{[T_{\mathsf{CW}}^{011}]^{\otimes \alpha N} \otimes [T_{\mathsf{CW}}^{101}]^{\otimes \alpha N} \otimes [T_{\mathsf{CW}}^{110}]^{\otimes \alpha N} \otimes [T_{\mathsf{CW}}^{002}]^{\otimes \left(\frac{1}{3}-\alpha\right)N} \otimes [T_{\mathsf{CW}}^{020}]^{\otimes \left(\frac{1}{3}-\alpha\right)N} \otimes [T_{\mathsf{CW}}^{200}]^{\otimes \left(\frac{1}{3}-\alpha\right)N}}.$$

$$T_{\mathsf{CW}}^{011} \cong \langle 1, 1, q \rangle$$
$$T_{\mathsf{CW}}^{101} \cong \langle q, 1, 1 \rangle$$
$$T_{\mathsf{CW}}^{110} \cong \langle 1, q, 1 \rangle$$

and

$$T_{\mathsf{CW}}^{002} \cong \langle 1, 1, 1 \rangle$$
$$T_{\mathsf{CW}}^{020} \cong \langle 1, 1, 1 \rangle$$
$$T_{\mathsf{CW}}^{200} \cong \langle 1, 1, 1 \rangle$$

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$$\cong \langle q^{\alpha N}, q^{\alpha N}, q^{\alpha N} \rangle$$

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Theorem 5

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$$\cong \langle q^{\alpha N}, q^{\alpha N}, q^{\alpha N} \rangle$$

Theorem 3 (the asymptotic sum inequality, general form) [Schönhage 1981]

$$\underline{R}\left(\bigoplus_{i=1}^{k} \langle m_i, n_i, p_i \rangle\right) \le t \Longrightarrow \sum_{i=1}^{k} (m_i n_i p_i)^{\omega/3} \le t$$

Analysis of the second construction

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$$2^{\left(H\left(\frac{2}{3}-\alpha,2\alpha,\frac{1}{3}-\alpha\right)-o(1)\right)N} \times q^{\alpha N \omega} \le \underline{R}(T_{\mathsf{CW}}^{\otimes N})$$

Analysis of the second construction

Theorem 5

For any $0 \leq \alpha \leq 1/3$ and for N large enough, the tensor $T_{\text{CW}}^{\otimes N}$ can be converted into a direct sum of

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$$[T_{\text{CW}}^{011}]^{\otimes \alpha N} \otimes [T_{\text{CW}}^{101}]^{\otimes \alpha N} \otimes [T_{\text{CW}}^{110}]^{\otimes \alpha N} \otimes [T_{\text{CW}}^{002}]^{\otimes \left(\frac{1}{3}-\alpha\right)N} \otimes [T_{\text{CW}}^{020}]^{\otimes \left(\frac{1}{3}-\alpha\right)N} \otimes [T_{\text{CW}}^{200}]^{\otimes \left(\frac{1}{3}-\alpha\right)N}.$$

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$$2^{\left(H\left(\frac{2}{3}-\alpha,2\alpha,\frac{1}{3}-\alpha\right)-o(1)\right)N} \times q^{\alpha N \omega} \leq \underline{R}(T_{\text{CW}}^{\otimes N}) \leq (q+2)^N$$

Analysis of the second construction

Theorem 5

For any $0 \leq \alpha \leq 1/3$ and for N large enough, the tensor $T_{\text{CW}}^{\otimes N}$ can be converted into a direct sum of

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$$\cong \langle q^{\alpha N}, q^{\alpha N}, q^{\alpha N} \rangle$$

Theorem 3 (the asymptotic sum inequality, general form) [Schönhage 1981]

$$\underline{R}\left(\bigoplus_{i=1}^{k} \langle m_i, n_i, p_i \rangle\right) \leq t \implies \sum_{i=1}^{k} (m_i n_i p_i)^{\omega/3} \leq t$$

$$2^{\left(H\left(\frac{2}{3}-\alpha,2\alpha,\frac{1}{3}-\alpha\right)-o(1)\right)N} \times q^{\alpha N \omega} \leq \underline{R}(T_{\text{CW}}^{\otimes N}) \leq (q+2)^N$$

$$\implies 2^{H\left(\frac{2}{3}-\alpha,2\alpha,\frac{1}{3}-\alpha\right)} \times q^{\alpha\omega} \leq (q+2)$$

Analysis of the second construction

Theorem 5

For any $0 \le \alpha \le 1/3$ and for N large enough, the tensor $T_{\mathsf{CW}}^{\otimes N}$ can be converted into a direct sum of

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$$\Longrightarrow 2^{H\left(\frac{2}{3}-\alpha,2\alpha,\frac{1}{3}-\alpha\right)} \times q^{\alpha\omega} \le (q+2)$$

$$\Longrightarrow \boxed{\omega \le 2.38718\ldots} \text{ for } q = 6 \text{ and } \alpha = 0.3173$$

Powers of the second construction (Section 5.3)

$$T_{\mathsf{CW}}^{\otimes 2} = (T_{\mathsf{CW}}^{011} + T_{\mathsf{CW}}^{101} + T_{\mathsf{CW}}^{110} + T_{\mathsf{CW}}^{002} + T_{\mathsf{CW}}^{020} + T_{\mathsf{CW}}^{200})^{\otimes 2}$$ (36 terms)

$\underline{R}(T_{\mathsf{CW}}^{\otimes 2}) \leq (q+2)^2$

Powers of the second construction (Section 5.3)

$$T_{\mathsf{CW}}^{\otimes 2} = (T_{\mathsf{CW}}^{011} + T_{\mathsf{CW}}^{101} + T_{\mathsf{CW}}^{110} + T_{\mathsf{CW}}^{002} + T_{\mathsf{CW}}^{020} + T_{\mathsf{CW}}^{200})^{\otimes 2} \quad \text{(36 terms)}$$

$\underline{R}(T_{\mathsf{CW}}^{\otimes 2}) \leq (q+2)^2$

$$\begin{aligned} T_{\mathsf{CW}}^{\otimes 2} = {} & T^{400} + T^{040} + T^{004} + T^{310} + T^{301} + T^{103} + T^{130} + T^{013} \\ & + T^{031} + T^{220} + T^{202} + T^{022} + T^{211} + T^{121} + T^{112}, \end{aligned}$$

where

$$\begin{aligned} T^{400} &= T_{\mathsf{CW}}^{200} \otimes T_{\mathsf{CW}}^{200}, \\ T^{310} &= T_{\mathsf{CW}}^{200} \otimes T_{\mathsf{CW}}^{110} + T_{\mathsf{CW}}^{110} \otimes T_{\mathsf{CW}}^{200}, \\ T^{220} &= T_{\mathsf{CW}}^{200} \otimes T_{\mathsf{CW}}^{020} + T_{\mathsf{CW}}^{020} \otimes T_{\mathsf{CW}}^{200} + T_{\mathsf{CW}}^{110} \otimes T_{\mathsf{CW}}^{110}, \\ T^{211} &= T_{\mathsf{CW}}^{200} \otimes T_{\mathsf{CW}}^{011} + T_{\mathsf{CW}}^{011} \otimes T_{\mathsf{CW}}^{200} + T_{\mathsf{CW}}^{110} \otimes T_{\mathsf{CW}}^{101} + T_{\mathsf{CW}}^{101} \otimes T_{\mathsf{CW}}^{110}, \end{aligned}$$

and the other 11 terms are obtained by permuting the variables (e.g., $T^{040} = T_{\mathsf{CW}}^{020} \otimes T_{\mathsf{CW}}^{020}$).

Analysis of the second power

Consider $\left[T_{\mathsf{CW}}^{\otimes 2}\right]^{\otimes N}$

Analysis of the second power

Consider $\left[T_{\mathsf{CW}}^{\otimes 2}\right]^{\otimes N}$

Theorem 6

For any $0 \leq \alpha, \beta, \gamma, \delta \leq 1$ such that $3\alpha + 6\beta + 3\gamma + 3\delta = 1$, the tensor $T_{\mathsf{CW}}^{\otimes 2N}$ can be converted into a direct sum of

$$2^{(H(2\alpha+2\beta+\gamma, 2\beta+2\delta, 2\gamma+\delta, 2\beta, \alpha)-o(1))N}$$

terms, each isomorphic to

$$\left[T^{400} \otimes T^{040} \otimes T^{004}\right]^{\otimes \alpha N} \otimes \left[T^{310} \otimes T^{301} \otimes T^{103} \otimes T^{130} \otimes T^{031} \otimes T^{013}\right]^{\otimes \beta N} \otimes \left[T^{220} \otimes T^{202} \otimes T^{022}\right]^{\otimes \gamma N} \otimes \left[T^{211} \otimes T^{121} \otimes T^{112}\right]^{\otimes \delta N}.$$

Analysis of the second power

Consider $\left[T_{\mathrm{CW}}^{\otimes 2}\right]^{\otimes N}$

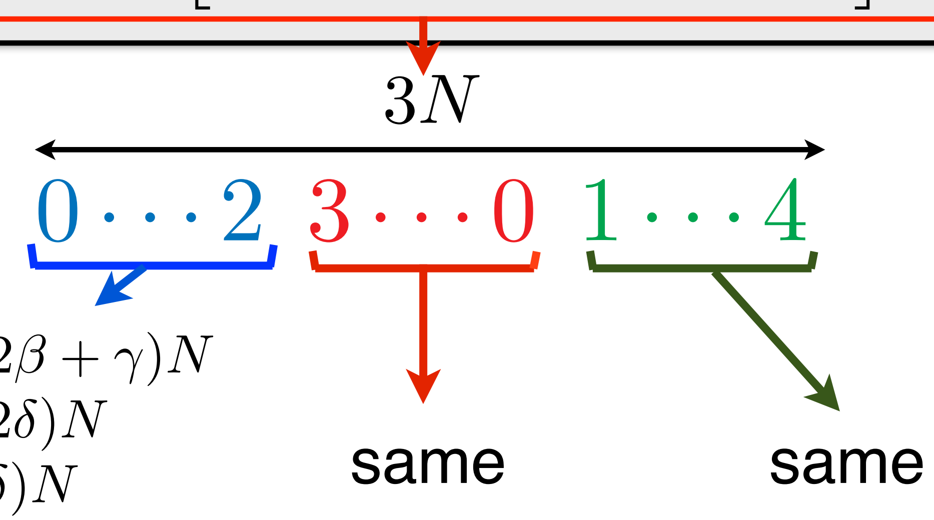
Theorem 6

For any $0 \le \alpha, \beta, \gamma, \delta \le 1$ such that $3\alpha + 6\beta + 3\gamma + 3\delta = 1$, the tensor $T_{\mathrm{CW}}^{\otimes 2N}$ can be converted into a direct sum of

$$2^{(H(2\alpha+2\beta+\gamma,2\beta+2\delta,2\gamma+\delta,2\beta,\alpha)-o(1))N}$$

terms, each isomorphic to

$$\left[T^{400} \otimes T^{040} \otimes T^{004}\right]^{\otimes \alpha N} \otimes \left[T^{310} \otimes T^{301} \otimes T^{103} \otimes T^{130} \otimes T^{031} \otimes T^{013}\right]^{\otimes \beta N}$$
$$\otimes \left[T^{220} \otimes T^{202} \otimes T^{022}\right]^{\otimes \gamma N} \otimes \left[T^{211} \otimes T^{121} \otimes T^{112}\right]^{\otimes \delta N}.$$



$\#0 = (2\alpha + 2\beta + \gamma)N$
$\#1 = (2\beta + 2\delta)N$
$\#2 = (2\gamma + \delta)N$
$\#3 = 2\beta N$
$\#4 = \alpha N$

Analysis of the second power

Consider $\left[T_{\mathsf{CW}}^{\otimes 2}\right]^{\otimes N}$

Theorem 6

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$3N$

$0 \cdots 2 \quad 3 \cdots 0 \quad 1 \cdots 4$

$\#0 = (2\alpha + 2\beta + \gamma)N$
$\#1 = (2\beta + 2\delta)N$
$\#2 = (2\gamma + \delta)N$
$\#3 = 2\beta N$
$\#4 = \alpha N$

same

same

number of possibilities

$$\binom{N}{\#0, \#1, \#2, \#3, \#4}$$

$$\approx 2^{H(2\alpha+2\beta+\gamma, 2\beta+2\delta, 2\gamma+\delta, 2\beta, \alpha)N}$$

Analysis of the second power

Consider $\left[T_{\mathsf{CW}}^{\otimes 2}\right]^{\otimes N}$

Theorem 6

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The asymptotic sum inequality gives:

$$2^{H(2\alpha+2\beta+\gamma, 2\beta+2\delta, 2\gamma+\delta, 2\beta, \alpha)} \times \boxed{} \le (q+2)^2$$

Analysis of the second power

Consider $\left[T_{\mathsf{CW}}^{\otimes 2}\right]^{\otimes N}$

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terms, each isomorphic to $\cong \langle c^N, c^N, c^N \rangle$

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The asymptotic sum inequality gives:

$$2^{H(2\alpha+2\beta+\gamma, 2\beta+2\delta, 2\gamma+\delta, 2\beta, \alpha)} \times c^{\omega} \leq (q+2)^2$$

Analysis of the second power

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Theorem 6

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The asymptotic sum inequality gives:

T^{211} need to be analyzed

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$\Longrightarrow$ $\omega \leq 2.375...$ for $q = 6$ and $\alpha = 0.00023$, $\beta = 0.0125$, $\gamma = 0.10254$ and $\delta = 0.2056$

History of the main improvements on the exponent of square matrix multiplication

Upper bound	Year	Authors
$\omega \leq 3$		
$\omega < 2.81$	1969	Strassen
$\omega < 2.79$	1979	Pan
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$\omega < 2.55$	1981	Schönhage
$\omega < 2.53$	1981	Pan
$\omega < 2.52$	1982	Romani
$\omega < 2.50$	1982	Coppersmith and Winograd
$\omega < 2.48$	1986	Strassen
$\omega < 2.376$	1987	Coppersmith and Winograd
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$\omega < 2.3729$	2012	Vassilevska Williams
$\omega < 2.3728639$	2014	Le Gall

Higher powers

T_{CW} gives $\omega < 2.388$

$T_{\mathsf{CW}}^{\otimes 2}$ gives $\omega < 2.376$

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What about higher powers?

Higher powers

T_{CW} gives $\omega < 2.388$

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What about higher powers?

Analyzing $T_{\mathsf{CW}}^{\otimes 3}$ was explicitly mentioned by Coppersmith and Winograd as an open question in 1990

Analysis of the first power

Theorem 5

For any $0 \leq \alpha \leq 1/3$ and for N large enough, the tensor $T_{\mathsf{CW}}^{\otimes N}$ can be converted into a direct sum of

$$2^{\left(H\left(\frac{2}{3}-\alpha,2\alpha,\frac{1}{3}-\alpha\right)-o(1)\right)N}$$

terms, each isomorphic to

$$\left[T_{\mathsf{CW}}^{011}\right]^{\otimes\alpha N}\otimes\left[T_{\mathsf{CW}}^{101}\right]^{\otimes\alpha N}\otimes\left[T_{\mathsf{CW}}^{110}\right]^{\otimes\alpha N}\otimes\left[T_{\mathsf{CW}}^{002}\right]^{\otimes\left(\frac{1}{3}-\alpha\right)N}\otimes\left[T_{\mathsf{CW}}^{020}\right]^{\otimes\left(\frac{1}{3}-\alpha\right)N}\otimes\left[T_{\mathsf{CW}}^{200}\right]^{\otimes\left(\frac{1}{3}-\alpha\right)N}.$$

$$\cong \langle q^{\alpha N}, q^{\alpha N}, q^{\alpha N}\rangle$$

$\Longrightarrow$ $\omega \leq 2.38718...$ for $q = 6$ and $\alpha = 0.3173$

Analysis of the first power

one variable

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Analysis of the second power

Consider $\left[T_{\mathsf{CW}}^{\otimes 2}\right]^{\otimes N}$

Theorem 6

For any $0 \leq \alpha, \beta, \gamma, \delta \leq 1$ such that $3\alpha + 6\beta + 3\gamma + 3\delta = 1$, the tensor $T_{\mathsf{CW}}^{\otimes 2N}$ can be converted into a direct sum of

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The asymptotic sum inequality gives:

T^{211} need to be analyzed

$$2^{H(2\alpha+2\beta+\gamma, 2\beta+2\delta, 2\gamma+\delta, 2\beta, \alpha)} \times \boxed{} \leq (q+2)^2$$

$\Longrightarrow \boxed{\omega \leq 2.375...}$ for $q = 6$ and $\alpha = 0.00023$, $\beta = 0.0125$, $\gamma = 0.10254$ and $\delta = 0.2056$

Analysis of the second power

Consider $\left[T_{\mathsf{CW}}^{\otimes 2}\right]^{\otimes N}$

3 variables

Theorem 6

For any $0 \leq \alpha, \beta, \gamma, \delta \leq 1$ such that $3\alpha + 6\beta + 3\gamma + 3\delta = 1$, the tensor $T_{\mathsf{CW}}^{\otimes 2N}$ can be converted into a direct sum of

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$$2^{H(2\alpha+2\beta+\gamma, 2\beta+2\delta, 2\gamma+\delta, 2\beta, \alpha)} \times \square \leq (q+2)^2$$

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Analysis of the second power

Consider $\left[T_{\mathsf{CW}}^{\otimes 2}\right]^{\otimes N}$ 3 variables

Theorem 6

For any $0 \leq \alpha, \beta, \gamma, \delta \leq 1$ such that $3\alpha + 6\beta + 3\gamma + 3\delta = 1$, the tensor $T_{\mathsf{CW}}^{\otimes 2N}$ can be converted into a direct sum of

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T_{CW} gives $\omega < 2.388$

$T_{\mathsf{CW}}^{\otimes 2}$ gives $\omega < 2.376$

What about higher powers?

Analyzing $T_{\mathsf{CW}}^{\otimes 3}$ was explicitly mentioned by Coppersmith and Winograd as an open question in 1990

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→ this does not (seem to) give any improvement

Overview of the Lectures

✓ Fundamental techniques for fast matrix multiplication (1969~1987)

- Basics of bilinear complexity theory: exponent of matrix multiplication, Strassen's algorithm, bilinear algorithms
- First technique: tensor rank and recursion
- Second technique: border rank
- Third technique: the asymptotic sum inequality

Lecture 1

- Fourth technique: the laser method

✓ Recent progress on matrix multiplication (1987~)

- Laser method on powers of tensors
- Other approaches
- Lower bounds

Lecture 2

- Rectangular matrix multiplication

✓ Applications of matrix multiplications, open problems

Lecture 3

Higher powers of the CW tensor

T_{CW} give

$T_{\text{CW}}^{\otimes 2}$ give

What abo

Analyzing

ope

The third

Analysis of the second power

Consider $[T_{\text{CW}}^{\otimes 2}]^{\otimes N}$

number of variables increases

Theorem 6

For any $0 < \alpha, \beta, \gamma, \delta \leq 1$ such that $3\alpha + 6\beta + 3\gamma + 3\delta = 1$, the tensor $T_{\text{CW}}^{\otimes 2N}$ can be converted into a direct sum of

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analysis of each “complex” component

(i.e., each component where 0 does not appear in the superscript)

Two difficulties when analyzing higher powers:

① how to analyze each component?

② the number of variables increases

(nonlinear conditions appear,

the optimization problem becomes non-convex)

Higher powers of the CW tensor

T_{CW} gives $\omega < 2.388$

$T_{\mathrm{CW}}^{\otimes 2}$ gives $\omega < 2.376$

What about higher powers?

Analyzing the third power was explicitly by mentioned as an open problem by Coppersmith and Winograd in 1990

The third power does not (seem to) give any improvement

But the fourth power actually does!

Two difficulties when analyzing higher powers:

① how to analyze each component?

② the number of variables increases
(nonlinear conditions appear,
the optimization problem becomes non-convex)

Higher powers of the CW tensor

Upper bounds on ω obtained by analyzing $T_{\mathrm{CW}}^{\otimes m}$

Difficulty ① solved using a recursive analysis

m	Upper bound	Number of variables in in the optimization problem	Authors
1	$\omega < 2.3871900$	1	CW (1987)
2	$\omega < 2.3754770$	3	CW (1987)
4	$\omega < 2.3729269$	9	Stothers (2010)
8	$\omega < 2.3729$	29	Vassilevska Williams (2012)
16	$\omega < 2.3728640$	101	LG (2014)
32	$\omega < 2.3728639$	373	LG (2014)

Difficulty ② solved: the optimization problem from the asymptotic sum inequality can be "relaxed" into a convex optimization problem

an efficient method to analyze the powers of the CW tensor, and any tensor "similar" to the CW tensor

History of the main improvements on the exponent of square matrix multiplication

Upper bound	Year	Authors
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$\omega < 2.3728639$	2014	LG

Rank of a tensor (1969–1979: Strassen, Pan)

Border rank of a tensor (1979: Bini, Capovani, Romani and Lotti)

Asymptotic sum inequality (1981–1982: Schönhage, Pan, Romani, Coppersmith and Winograd)

Laser method (1986–2014: Strassen, Coppersmith and Winograd, Stothers, Vassilevska Williams, LG)

What is ω? $\omega = 2$?

Limitations of the Laser Method on the CW tensor

Upper bounds on ω obtained by analyzing $T_{\mathrm{CW}}^{\otimes m}$

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The same result holds for what can be obtained from the tensor T_{CW} by applying all the possible variants suggested in Coppersmith and Winograd's original paper

Can this analysis (for powers 64, 128,...) converge to 2?

First result in [Ambainis, Filmus and LG, 2015]

No, the same analysis for these powers cannot show ω < 2.3725

Can the analysis of any power of T_{CW} using the laser method converge to 2?

Second result in [Ambainis, Filmus and LG, 2015]

No, such analyses cannot show ω < 2.3078

Powers of the CW tensor: Conclusions

- ✓ Progress on the exponent of matrix multiplication has been done in the last seven years by analyzing powers of the CW tensor
- ✓ A new tensor is probably needed to make any further significant improvement on ω
- ✓ We now have efficient methods to analyze the powers of tensors that have a structure similar to the structure of the CW tensor
- ✓ Unfortunately, we currently do not have any other good tensor to analyze with these methods

Hope: obtain new tensors using a group-theoretic approach

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Lecture 3

The Group Theoretic Approach

[Cohn, Umans 2003]
[Cohn et al. 2005]

- ✓ Consider a finite group *G* and its group algebra $\mathbb{C}[G]$

  elements of the group algebra are formal sums of the elements of the group

$$\sum_{g \in G} \alpha_g \boldsymbol{g} \quad \text{with } \alpha_g \in \mathbb{C}$$

- ✓ The group algebra is isomorphic to a direct product of matrix products

$$\mathbb{C}[G] \cong \mathbb{C}^{d_1 \times d_1} \times \mathbb{C}^{d_2 \times d_2} \times \cdots \times \mathbb{C}^{d_k \times d_k}$$

  where $d_1, d_2, \ldots, d_k$ are the dimensions of the irreducible representations of *G*

$$d_1^2 + d_2^2 + \cdots + d_k^2 = |\mathrm{G}|$$

- ✓ Multiplication in $\mathbb{C}[G]$, i.e., multiplication of two formal sums, can thus be done in time roughly $d_1^\omega + d_2^\omega + \cdots + d_k^\omega$

The Group Theoretic Approach

[Cohn, Umans 2003]
[Cohn et al. 2005]

Key definition ([Cohn, Umans 2003])

The group G realizes ⟨m,m,m⟩ if there exist three sets S,T,U ⊆ G such that:

(i) |S|=|T|=|U|=m, and

(ii) S, T, U satisfy the triple product property.

- ✓ Then consider two n x n real matrices A and B
- ✓ Consider the two elements $\sum_{s\in S, t\in T} a_{st}\, \boldsymbol{st}$ and $\sum_{t\in T, u\in U} b_{tu}\, \boldsymbol{tu}$ of $\mathbb{C}[G]$
- ✓ Their product is $\sum_{s\in S\; u\in U} \left(\sum_{t\in T} a_{st} b_{tu} \right) \boldsymbol{su}$ and the $\boldsymbol{su}$ are distinct

  (from the triple product property)

If G realizes ⟨m,m,m⟩ then the product of two complex m x m matrices can be extracted from one product in $\mathbb{C}[G]$.

The Group Theoretic Approach

[Cohn, Umans 2003]
[Cohn et al. 2005]

If G realizes ⟨m,m,m⟩ then the product of two real m x m matrices can be extracted from one product in $\mathbb{C}[G]$.

$$\mathbb{C}[G] \cong \mathbb{C}^{d_1 \times d_1} \times \mathbb{C}^{d_2 \times d_2} \times \cdots \times \mathbb{C}^{d_k \times d_k}$$

$d_1, d_2, \ldots, d_k$: dimensions of the irreducible representations of G

Multiplication in $\mathbb{C}[G]$, i.e., multiplication of two formal sums, can thus be done in time roughly $d_1^\omega + d_2^\omega + \cdots + d_k^\omega$

Conclusion:
$$\mathrm{m}^\omega \le d_1^\omega + d_2^\omega + \cdots + d_k^\omega$$

GOAL

To obtain a good bound on ω using this framework, find a group that realizes a large matrix product and has irreducible representations of small dimensions

The Group Theoretic Approach

[Cohn, Umans 2003]
[Cohn et al. 2005]

[Cohn, Umans 2003]

establishes this framework, prove some conditions on the group that would lead to ω = 2, but not able to find a group leading to any non-trivial bound ω < 3

[Cohn, Kleinberg, Szegedy, Umans 2005]

finds the first explicit group leading to ω < 3, shows how to recover Coppersmith and Winograd's bound ω < 2.376 in this framework, and present a conjecture ("the strong Uniquely Solvable Puzzle conjecture") that would show ω = 2

[Alon, Shpilka, Umans 2013]

shows that the strong Uniquely Solvable Puzzle conjecture contradicts a (multicolored version) of Erdös-Szemerédi sunflower conjecture

[Blasiak et al. 2016]

proves the (multicolored version) of Erdös-Szemerédi sunflower conjecture and thus disproves the strong Uniquely Solvable Puzzle conjecture

extending recent breakthrough results on cap sets by Croot, Lev, Pach, Ellenberg and Gijswit

GOAL:
To obtain a good bound on ω using this framework, find a group that realizes a large matrix product and has irreducible representations of small dimensions

The Group Theoretic Approach

[Cohn, Umans 2003]
[Cohn et al. 2005]

The group theoretic approach still appears as a valid approach to find new constructions that may lead to further progress on ω

[Cohn, Umans 13]

Embedding of matrix multiplication into more general algebras (can again recover Coppersmith and Winograd's bound $\omega < 2.376$ in this framework)

GOAL

To obtain a good bound on ω using this framework, find a group that realizes a large matrix product and has irreducible representations of small dimensions