Special topic: Almost-fine-grained Complexity

From Gap-ETH to FPT Inapproximability: Clique, Dominating Set and More

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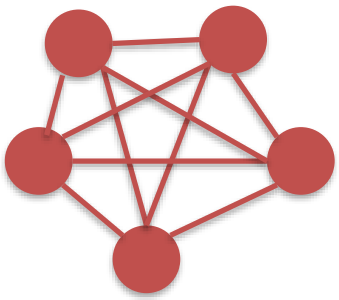
The problems

The k-clique problem

- **Input**: **n**-vertex undirected graph $G = (V, E)$
- **Output**: A clique of size **k**



The k-clique problem

- **Input**: **n**-vertex undirected graph $G = (V, E)$
- **Output**: A clique of size **k**

A trivial algorithm:

n^k

Enumerate all k-subsets of vertices and check whether it's a clique

$n^{\omega k/3}$

A not-so-trivial improvement:

Reduction to Boolean matrix multiplication

"Enumerative" running time: $n^{\Theta(k)}$ where
k = parameter

Beyond Enumerative Running Time?

- **Unlikely for k-Clique**
  Any $n^{o(k)}$ algorithm would imply $2^{o(n)}$ algorithm for solving 3SAT

Breaking Exponential Time Hypothesis (ETH)

What about approximation algorithms?

- **Input**: $G = (V, E)$
- **Promise**: There is a clique of size **q**
- **Output**: A clique of size **k**

(q,k)-Clique

A simple trick: If there is a clique of size ϵn, possible to beat $n^{o(k)}$

- **Partition** $V = V_1 \cup V_2 \cup \cdots \cup V_{\frac{n}{\log n}}$
- **Return** $\max_j clique(G[V_j])$

at least $\epsilon \log n$

Runtime: poly(n)

Open question: Is there a function F for which $(F(k), k)$-Clique is $n^{o(k)}$ solvable?

Think $F(k) = 2^{2^k}$

K-Dominating Set

- **Input**: $G = (V, E)$
- **Promise**: There is a dominating set of size **k**
- **Output**: A dominating set of size **q**

(k,q)-DomSet

Seems much harder than cliques

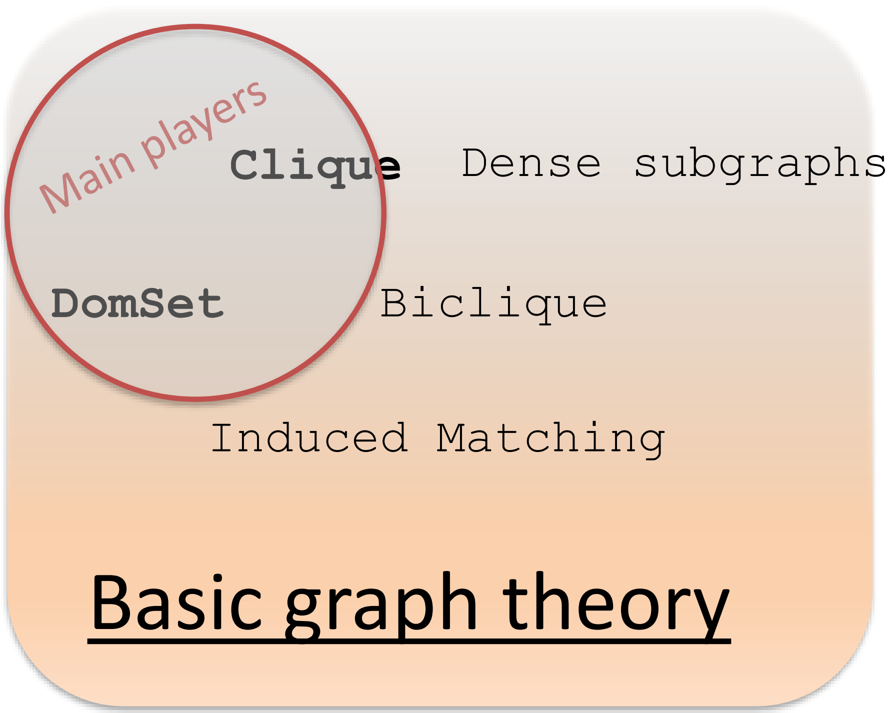
- Solvable exactly in $\boldsymbol{n^{(1+o(1))k}}$ time **[Patrascu-Williams 08]**
- In time $\boldsymbol{n^{k-\epsilon}}$, nothing beyond $(\boldsymbol{k}, \boldsymbol{k} \log \boldsymbol{n})$ −**DomSet**

Open question: Is there a function F for which $(k, F(k))$-DomSet is $n^{o(k)}$ solvable?

Think $F(k) = 2^{2^k}$

Breaking enumeration-type running time for optimization problems?

Generic Problems: [E.g. **Π=maximization problem**]
Find a size-**k** solution for problem Π given promise of size-**q** solution



- Solvable Exactly in $n^{\Theta(k)}$ time
- No non-trivial $(F(k), k) - \Pi$ algorithm in $n^{o(k)}$

Think $F(k) = 2^{2^{2^k}}$

OUR RESULTS

Many aforementioned problems are inherently enumerative

Consequence: FPT Inapproximability

(don't need to remember)

- k= parameter
- $\alpha(k)$-approximation Algorithm
- Running time $t(k)poly(n)$

Key open problems:

Non-trivial FPT approximation for **Clique** or **DomSet**?

- **Clique** is *FPT-inapproximable* if
  - No $o(k)$ approximation in time $t(k)poly(n)$

Our setting: $n^{o(k)}$

Fact:

No improvement over enumerative running time ⇒ No non-trivial FPT approximation

Our Complexity Assumption:

Gap-ETH

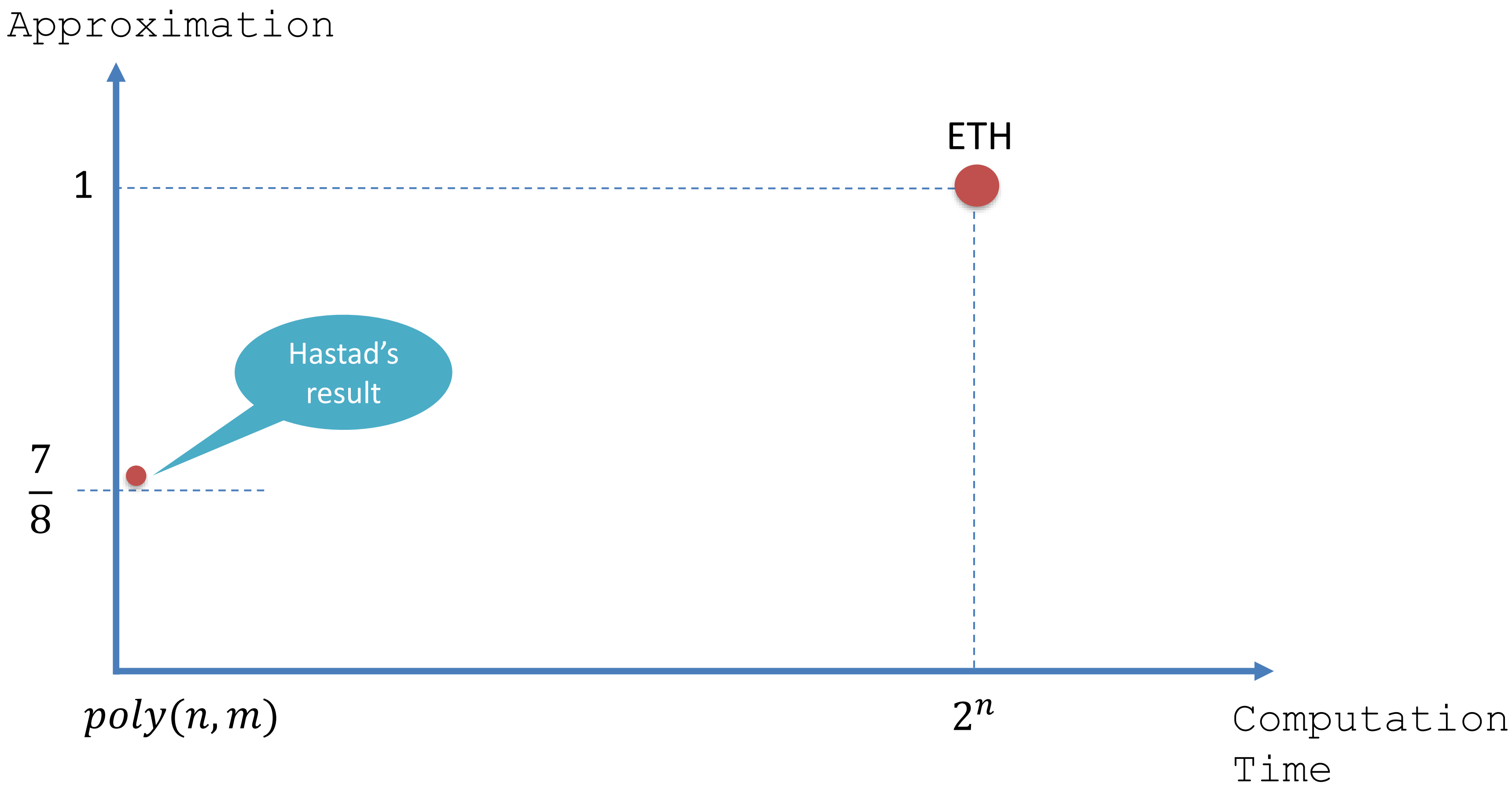
Exponential Time Hypothesis

ETH:
3SAT cannot be decided in time $2^{o(n)}$ or $2^{o(m)}$

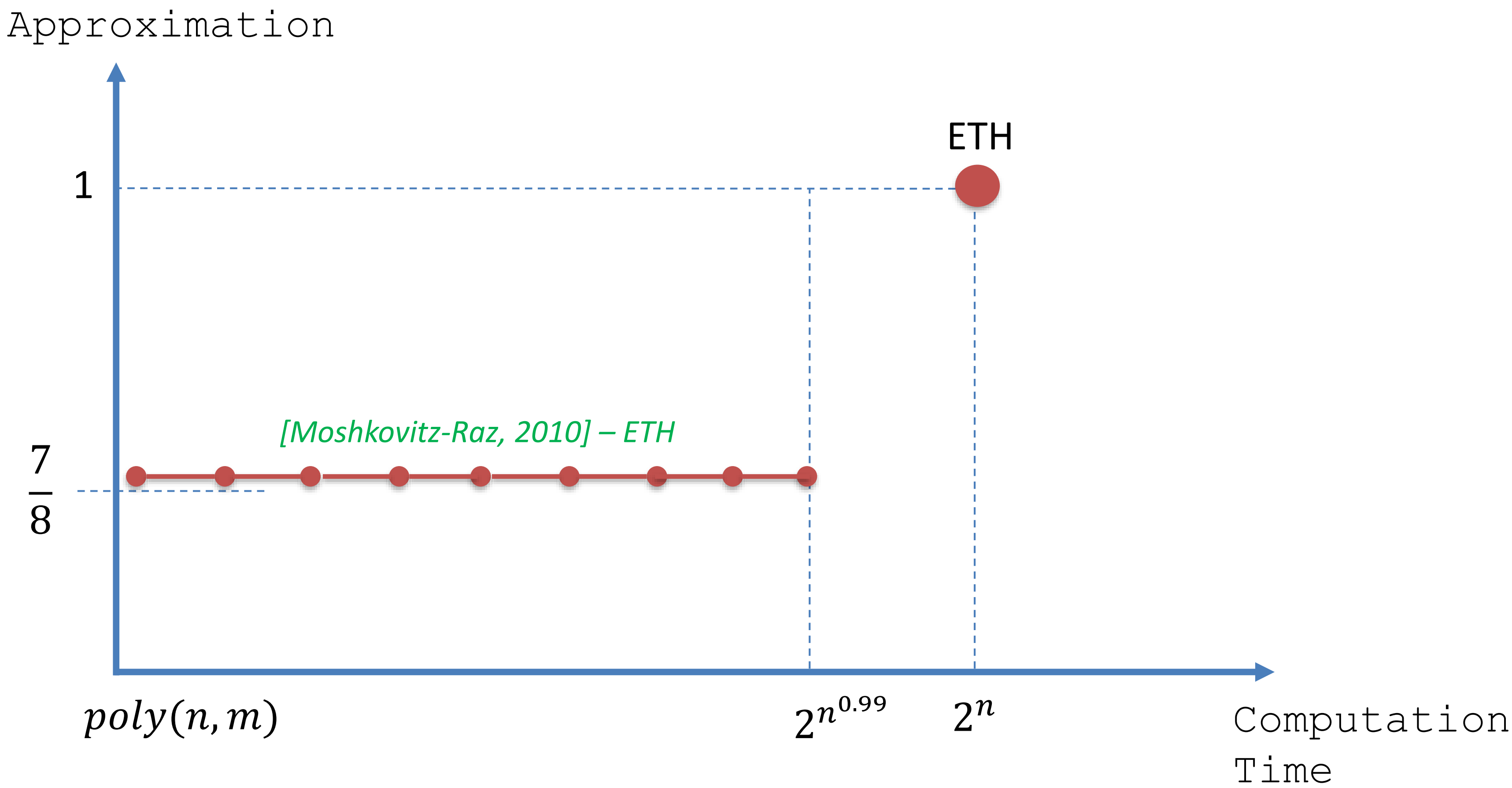
Used for ruling out $n^{o(k)}$ for k-clique

Therefore, we need at least an assumption as strong as ETH to study our question

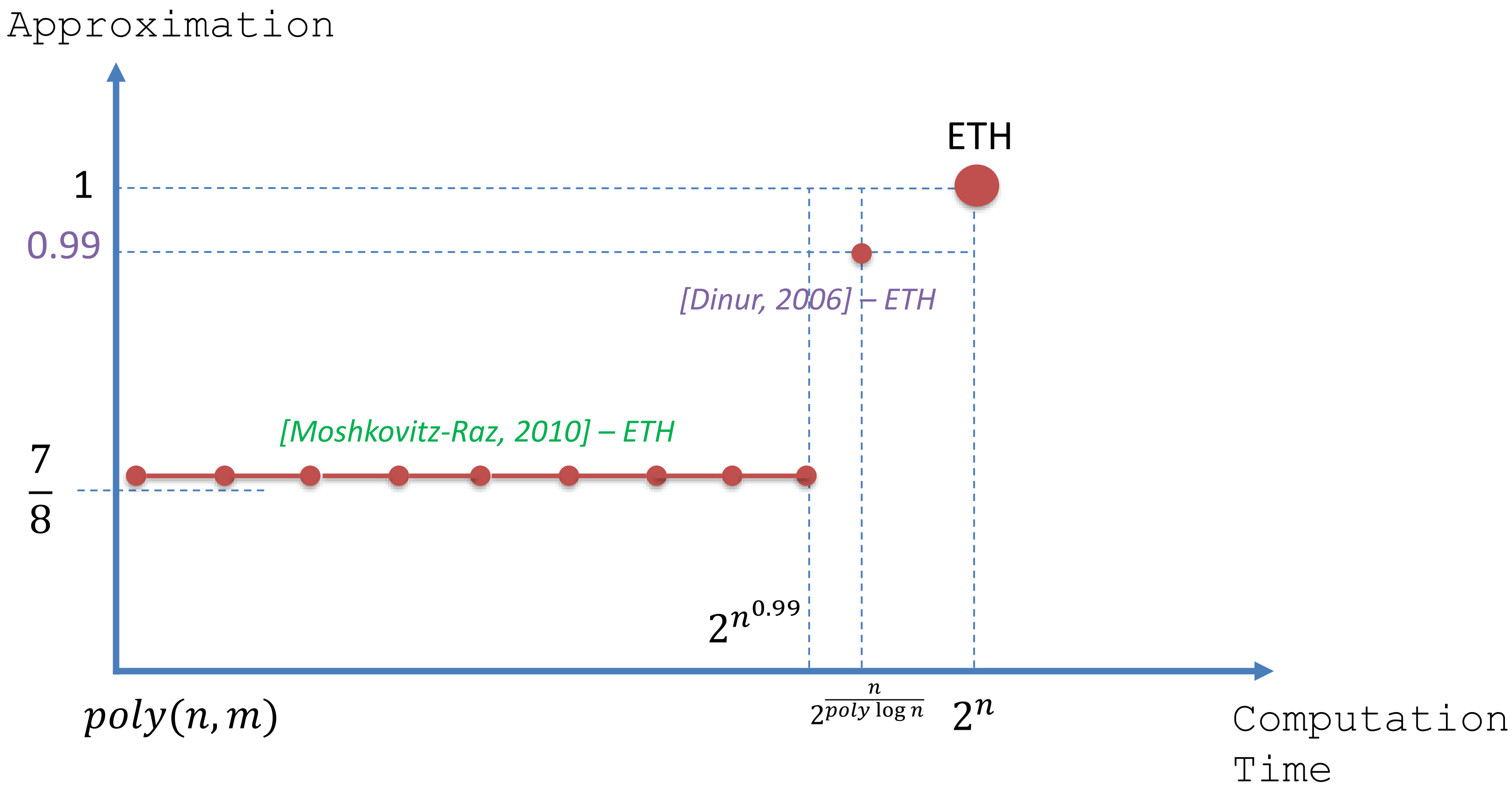
A fine-grained complexity of 3SAT



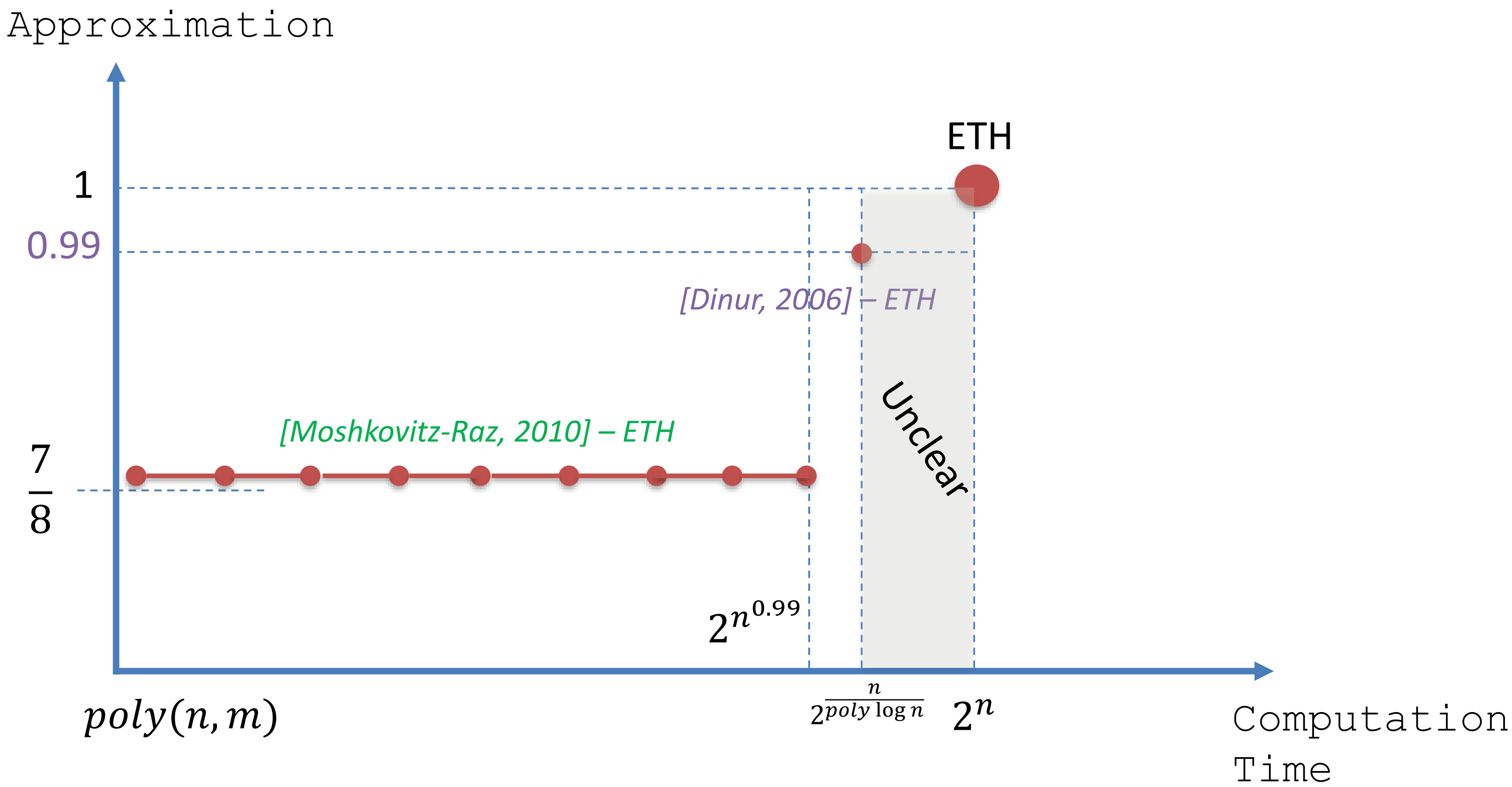
A fine-grained complexity of 3SAT



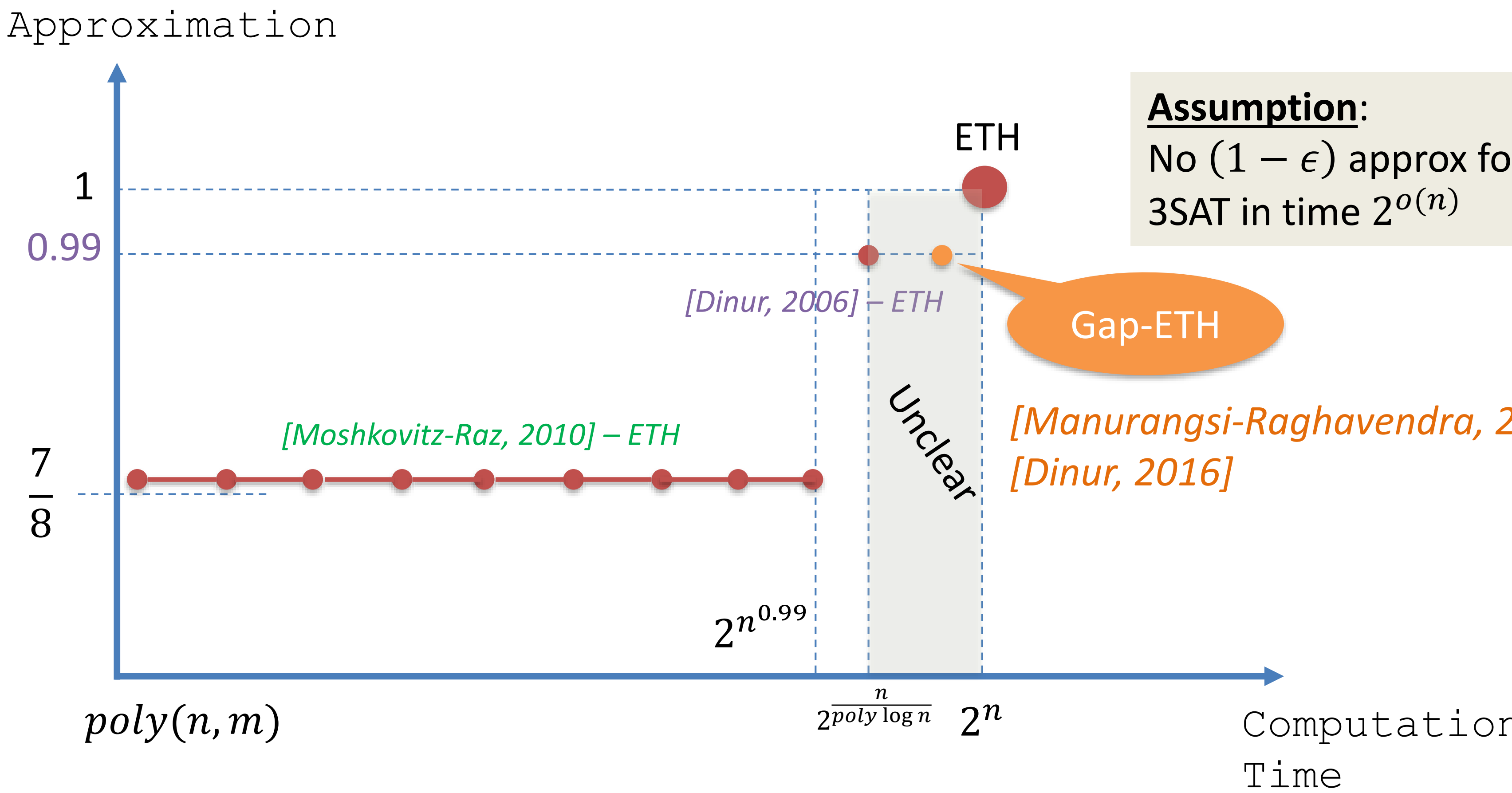
A fine-grained complexity of 3SAT



A fine-grained complexity of 3SAT



A fine-grained complexity of 3SAT



Assumption:
No $(1-\epsilon)$ approx for 3SAT in time $2^{o(n)}$

[Manurangsi-Raghavendra, 2016]
[Dinur, 2016]

Gap-ETH Lower Bounds

- **Inherently enumerative problems**
  - Clique
  - Dominating Set
  - Bipartite Induced Matching
  - Biclique

Knowing existence of $2^{2^{2^k}}$-clique does not help finding k-clique

- **Weakly inherently enumerative problems**
  - Max Induced Subgraph with hereditary properties

Hey, if you don't know what it is, ignore it!

Technique: A reduction from optimization problems on Label Cover instance

A showcase: Clique

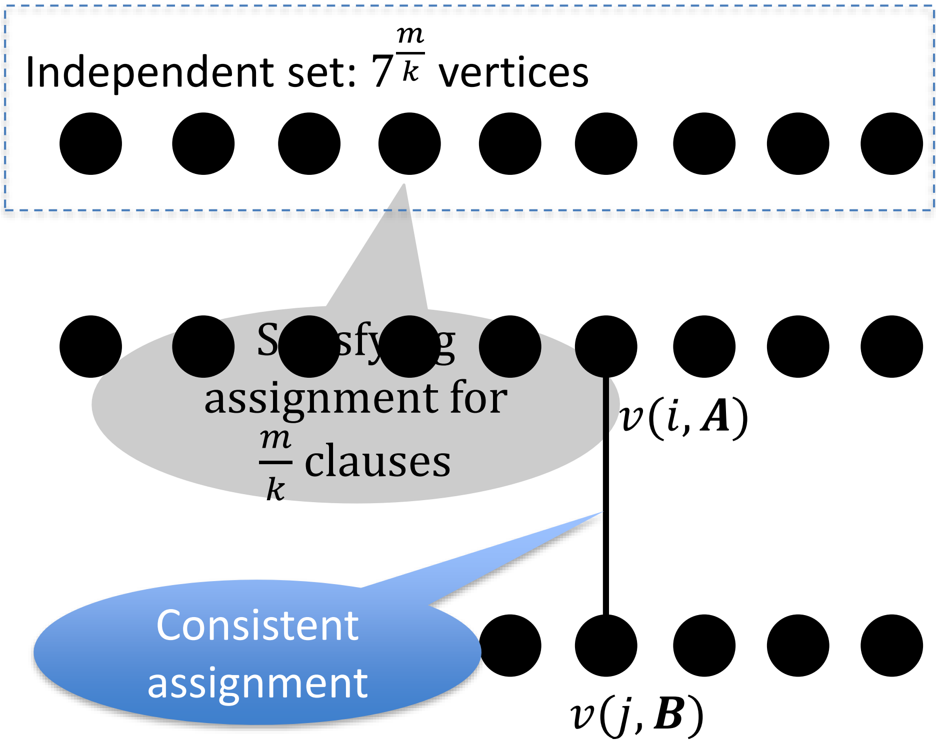
A warm-up: ETH-hardness of k-clique

$$\bigwedge_{i=1\ldots m} C_i \Rightarrow \bigwedge_{i=1\ldots\frac{m}{k}} C_i$$

$\Rightarrow$

(k groups)

$$\Rightarrow \bigwedge_{i=(m-\frac{m}{k})\ldots m} C_i$$



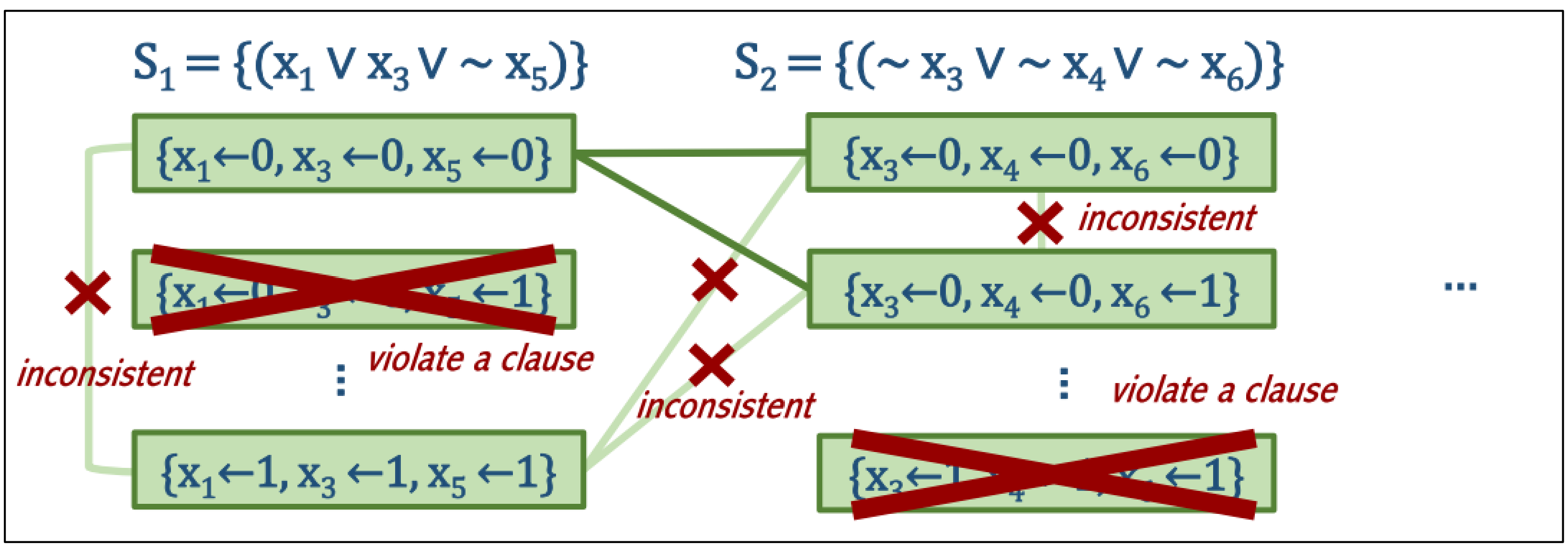
Step 1: Partitioning 3SAT formula into k groups

Step 2: Create a graph

$$|V(G)| = k7^{m/k}$$

Example (with $|I_j| = 1$)

$(x_1 \lor x_3 \lor \sim x_5)$
$\land\ (\sim x_3 \lor \sim x_4 \lor \sim x_6)$
$\land\ (x_2 \lor \sim x_4 \lor x_6)$
$\land\ \ldots$



Compression

- **Reduction from 3SAT to CLIQUE**
  - **<u>Size</u>**: $\mathrm{N} = \; k\, 2^{O\left(\frac{m}{k}\right)}$
  - **<u>Solutions</u>**: Clique of size **k** if and only if 3SAT formula is satisfiable

<u>ETH-hardness</u>:

Solving k-Clique in time $\left(\frac{N}{k}\right)^{o(k)}$ implies deciding 3SAT in time $2^{o(m)}$

Gap-ETH Hardness

$$\bigwedge_{i=1\ldots m} C_i$$

Random I_1

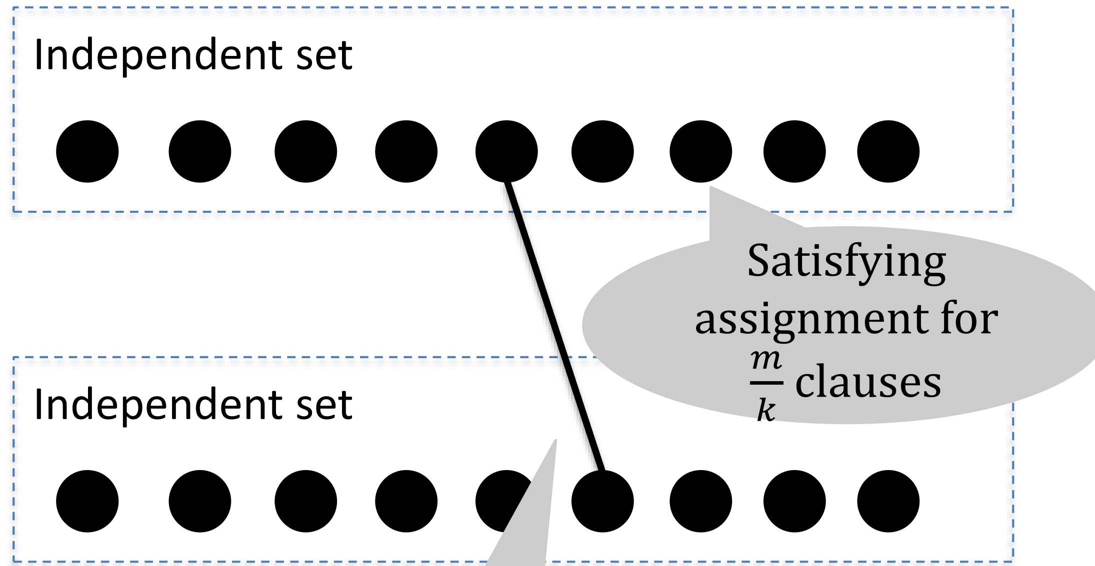
$|I_1| = \frac{m}{\epsilon k}$

$$\bigwedge_{j\in I_1} C_j$$

Random I_2

$$\bigwedge_{j\in I_2} C_j$$

Keep doing this for q rounds



Analysis

- **Completeness**: If 3SAT has satisfying assignment, then there is a **q**-clique

Soundness:

The probability of having a clique of size 100k is very low … (exercises)

Concluding remarks

Take-home message

ETH → Exact lower bound in time $n^{o(k)}$

Gap-ETH → Approximation lower bound in time $n^{o(k)}$

Our work!

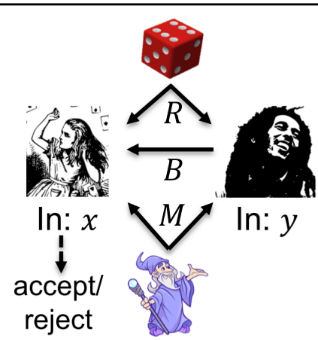
Follow-up: Dominating Set

[C.S., Manurangsi, Laekhanukit, 2018]

Hypothesis	Inapprox	Running Time	
W[1]≠FPT	$\Omega(\log^{1/poly(k)} n)$	FPT-Time (T(k) poly(n))	
ETH	$\Omega(\log^{1/poly(k)} n)$	$n^{o(k)}$	
SETH	$\Omega(\log^{1/f(k)} n)$	$n^{k-\varepsilon}$	for any $\varepsilon > 0$
k-SUM	$\Omega(\log^{1/poly(k)} n)$	$n^{k/2-\varepsilon}$	for any $\varepsilon > 0$

- SETH → No $(\log n)^{\frac{1}{poly(k)}}$ approximation for k-DomSet in time $n^{k-\epsilon}$
- ETH → Inherently enumerative
- $W[1] \neq FPT$ → FPT-inapproximability

Tool from fine-grained complexity:
"Distributed PCP" [Abboud, Rubinstein, Williams'17]
(See Karl's lecture)



Thank you!