# ADFOCS 2020: Fair Division

## Problem Set 3

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1. Assume that a set $M$ of $m$ goods need to be fairly divided among a set $N$ of $n$ agents, and each agent has a monotone subadditive valuation function $v$, where $v(S \cup T) \leq v(S) + v(T), \forall S, T \subseteq M$ and $v(S) \leq v(T), \forall S \subseteq T \subseteq M$. Show that

   *a.* Envy-freeness (EF) implies proportionality (Prop).

   *b.* Prop does not imply EF.

2. Show all Pareto optimal (PO) allocations in the following example with 2 agents $\{a_1, a_2\}$ with additive valuations and 4 indivisible goods $\{g_1, g_2, g_3, g_4\}$, where the value of each good for each agent is given as follows:

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| $a_1$ | 3 | 1 | 8 | 0 |
| $a_2$ | 5 | 0 | 7 | 3 |

   How many are EF1? How many are EFX?

3. Show that the envy-cycle procedure runs in polynomial time.

4. *a.* Show an example with additive valuations for which the envy-cycle procedure does not give an EFX allocation.

   *b.* Show that EFX allocations exist when agents have identical general monotone valuations.

   *c.* Design a polynomial-time algorithm to obtain an EFX allocation when agents have identical additive valuations.

   *d.* Design a polynomial-time algorithm to obtain an EFX allocation when there are two agents with additive valuations.

5. An allocation $A = (A_1, \ldots, A_n)$ is called $\alpha$-EFX if

$$v_i(A_i) \geq \alpha \cdot v_i(A_j \setminus g), \ \forall g \in A_j, \ \forall i, j \ .$$

   Design a polynomial-time algorithm to obtain $\frac{1}{2}$-EFX allocation when agents have monotone subadditive valuations. [Hint: envy-cycle procedure]

6. Suppose we want to fairly allocate a set $M$ of indivisible *chores*, for which each agent has negative utility, i.e., $v_i(S) < 0, \forall S \subseteq M, \forall i$. What is the natural analogue of EF1 allocation in this case? Design a polynomial-time algorithm to obtain an EF1 allocation when agents have additive valuations.

7. Suppose the set of indivisible items consists of both goods and chores. What is the natural analogue of EF1 allocation in this case? Design a polynomial-time algorithm to obtain an EF1 allocation when agents have additive valuations.