Spectral Graph Theory

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Outline

Convex Optimization

Convex Programs

Definition

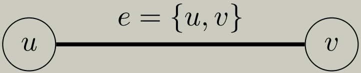
Given a convex set $K \subseteq \mathbb{R}^n$ and a convex $f : K \to \mathbb{R}$, a convex program is the following optimization problem

$$\inf_{x \in K} f(x).$$

Graphs

Undirected Graph

$G = (V, E),\ E \subseteq \binom{V}{2}$



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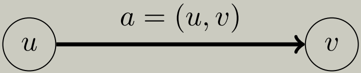
$G = (V, A),\ A \subseteq V \times V$



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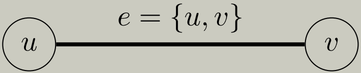

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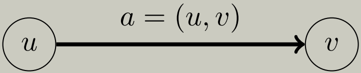

Node-Arc-Incidence Matrix $B \in \{-1, 0, 1\}^{|V| \times |A|}$

$$B_{v,a} = \begin{cases} -1 & \text{if } a = (v, u) \\ 1 & \text{if } a = (u, v) \\ 0 & \text{otherwise} \end{cases}$$

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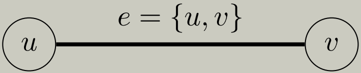

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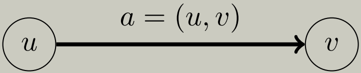

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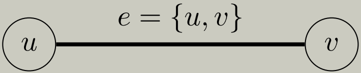

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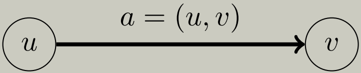

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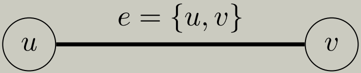

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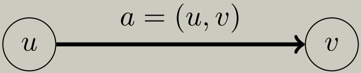

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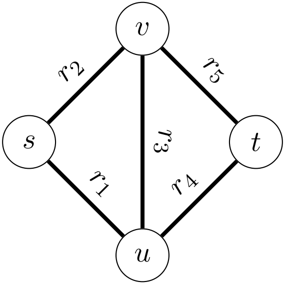
- $\mathbb{1}^T B = 0$, $rank(B) = |V| - 1$
- $\mathbb{1}_v^T B f = \sum_{a=(u,v)} f(a) - \sum_{a=(v,u)} f(a) = f(\delta^{in}(v)) - f(\delta^{out}(v))$

Electrical Flow Problem



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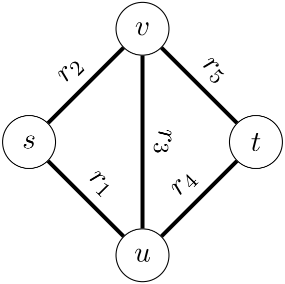

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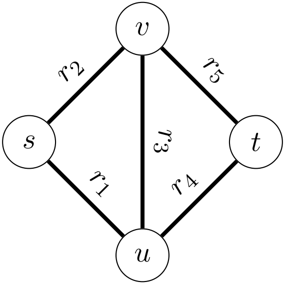

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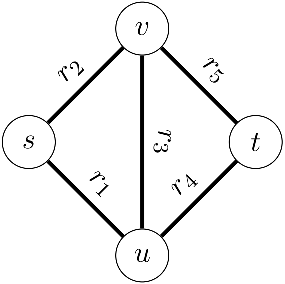

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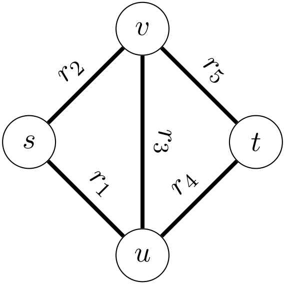

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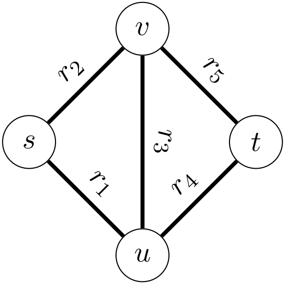

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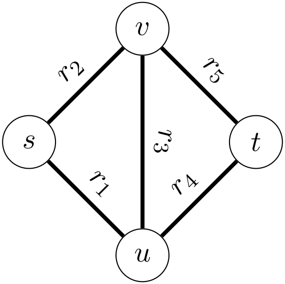

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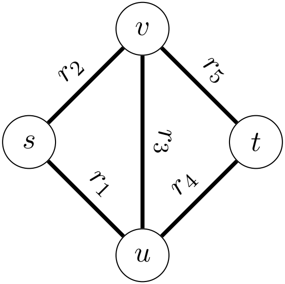

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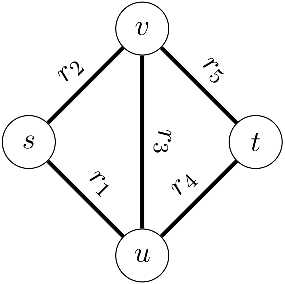

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$$Lp = d$$

Properties of a Laplacian $L := BWB^T$

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$$L_{uv} = \begin{cases} -w(e) & \text{for } e = \{u, v\} \in E \\ \sum_{e \in \delta(v)} w(e) & \text{for } u = v \\ 0 & \text{otherwise} \end{cases}$$

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Pseudo-Inverse

Moore-Penrose Inverse of a Laplacian

Let $Lu_i = \lambda u_i$ for $i \in [n]$ with $u_i^T u_j = \delta_{ij}$ and $\lambda_1 \leq \cdots \leq \lambda_n$. The matrix

$$L^+ := \sum_{i=2}^{n} \frac{1}{\lambda_i} u_i u_i^T$$

satisfies the Moore-Penrose conditions:

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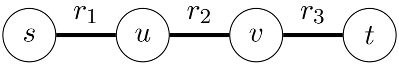
Simple Example: Path



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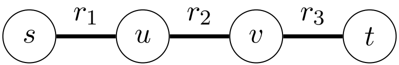

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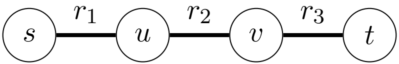
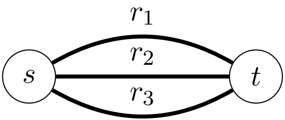


- $Bf = \mathbb{1}_t - \mathbb{1}_s$ has the unique solution $\hat{f} = \mathbb{1}$ with objective value $tr(R)$.
- This is equivalent to contracting the path to a single edge with resistance $tr(R)$, which is called the *effective resistance* between s and t.

Effective Resistance of Parallel Edges



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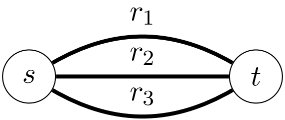

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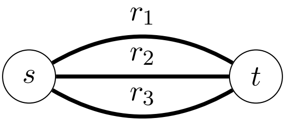

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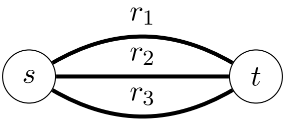

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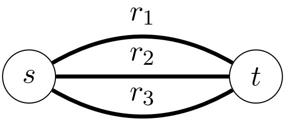
- $\sum_{i=1}^{m} r_i \hat{f}_i^2 = \sum_{i=1}^{m} \frac{(\hat{p}_t - \hat{p}_s)^2}{r_i}$

Effective Resistance of Parallel Edges



- $r_i \hat{f}_i = \hat{p}_t - \hat{p}_s$ for all $i \in [m]$
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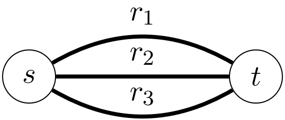

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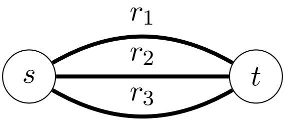

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- This is equivalent to substituting the parallel edges by a single edge with effective resistance $R_{eff}(s,t)$ where $\frac{1}{R_{eff}(s,t)} = \sum_{i=1}^{m} \frac{1}{r_i}$.

Effective Resistances

$$R_{eff}(s,t) := \min\{f^T R f : Bf = \mathbb{1}_t - \mathbb{1}_s\}$$

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R_{eff} is a distance

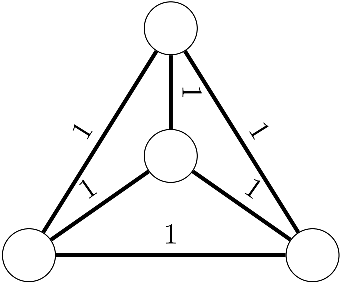
1. $R_{eff}(s,t) \geq 0$
2. $R_{eff}(s,t) = 0 \Leftrightarrow s = t$
3. $R_{eff}(s,t) = R_{eff}(t,s)$
4. $R_{eff}(s,t) \leq R_{eff}(s,v) + R_{eff}(v,t)$

Eigenvalue Bounds for the Electrical Energy

Let $d \in \mathbb{R}^n$ with $\mathbb{1}^T d = 0$ and $\hat{p} = L^+ d$

$$\frac{||d||_2^2}{\lambda_n} \leq \hat{p}^T L \hat{p} \leq \frac{||d||_2^2}{\lambda_2}$$

Example: Unweighted Complete Graph



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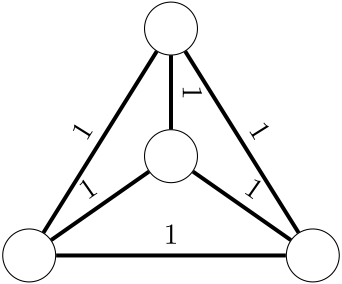

Laplacian

$$L = nI - \mathbb{1}\mathbb{1}^T$$

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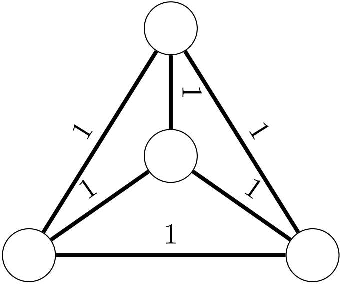

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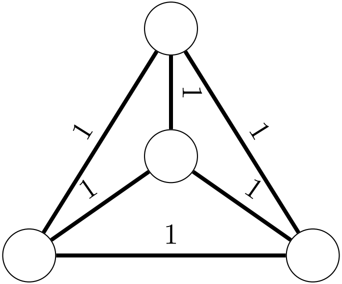

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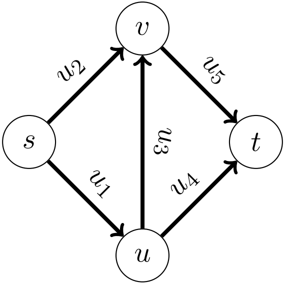
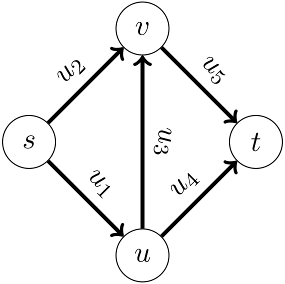
Energy dissipation of d is $||d||_2^2/n$

Max-Flow



Max-Flow

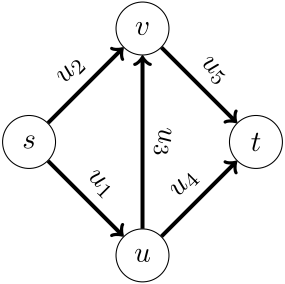


Directed Graph $G = (V, A)$, distinct $s, t \in v$, $u \in \mathbb{R}^m_{>0}$

Augment G by an arc (t, s) with infinite capacity to obtain $\hat{G}$.

$$F := \max\{\hat{f}(t,s) : \hat{B}\hat{f} = 0,\ 0 \leq \hat{f} \leq \hat{u}\}$$

Max-Flow



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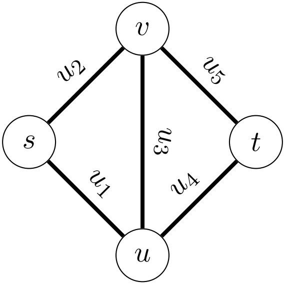
Can be solved by Interior Point Methods

using electrical flow as a subroutine.

Undirected Max-Flow



Undirected Graph $G = (V, E)$, distinct $s, t \in v$, $u \in \mathbb{R}^m_{>0}$

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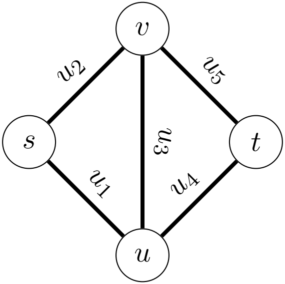
Undirected Max-Flow



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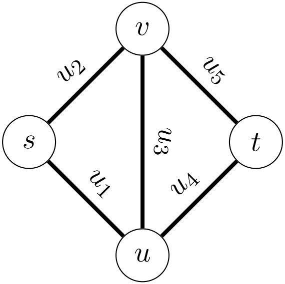

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\end{aligned}
$$

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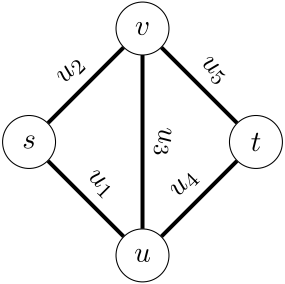

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Reciprocal problem: Congestion Minimization

$$1/F = \min\{||U^{-1}f||_\infty : Bf = \mathbb{1}_{(s,t)}\}$$