ADFOCS22	Summer 2021

Solving Laplacian Linear Equations

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Exercise 1.

Let $\boldsymbol{L}$ be the Laplacian of a connected, weighted, undirected graph G, and let $\boldsymbol{B} \in \mathbb{R}^E$ be the associated edge-vertex incidence matrix. Let $\boldsymbol{d} \in \mathbb{R}^V, \boldsymbol{d} \perp \boldsymbol{1}$ be a demand vector. Assume we are given $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{f}}$ such that

$$\boldsymbol{L}\tilde{\boldsymbol{x}} = \boldsymbol{d} \text{ and } \tilde{\boldsymbol{f}} = \boldsymbol{R}^{-1}\boldsymbol{B}^\top \tilde{\boldsymbol{x}}$$

1. Prove that $\tilde{\boldsymbol{x}}^\top \boldsymbol{L}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{f}}^\top \boldsymbol{R}\tilde{\boldsymbol{f}} = \boldsymbol{d}^\top \boldsymbol{L}^+ \boldsymbol{d}$.

Exercise 2.

Let $\boldsymbol{M} = \boldsymbol{X}\boldsymbol{Y}\boldsymbol{X}^\top$ for some $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n\times n}$, where $\boldsymbol{X}$ is invertible and $\boldsymbol{M}$ is symmetric. Furthermore, consider the spectral decomposition of $\boldsymbol{M} = \sum_{i=1}^n \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^\top$. Then, we define $\boldsymbol{\Pi}_{\boldsymbol{M}} = \sum_{i,\lambda_i \neq 0} \boldsymbol{v}_i \boldsymbol{v}_i^\top$. $\boldsymbol{\Pi}_{\boldsymbol{M}}$ is the orthogonal projection onto the image of $\boldsymbol{M}$: It has the property that for $\boldsymbol{v} \in \operatorname{im}(\boldsymbol{M})$, $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{v}$ and for $\boldsymbol{v} \in \ker(\boldsymbol{M})$, $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{0}$.

Prove that

$$\boldsymbol{Z} = \boldsymbol{\Pi}_{\boldsymbol{M}}(\boldsymbol{X}^\top)^{-1}\boldsymbol{Y}^+\boldsymbol{X}^{-1}\boldsymbol{\Pi}_{\boldsymbol{M}}$$

is the pseudoinverse of $\boldsymbol{M}$.

Exercise 3

1. Prove that for if $\boldsymbol{A} \preceq \boldsymbol{B}$, where both are $n \times n$, then for any real matrix $\boldsymbol{C} \in \mathbb{R}^{d\times n}$, $\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top \preceq \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top$.
2. Prove that if $\boldsymbol{0} \prec \boldsymbol{A} \preceq \boldsymbol{B}$ then $\boldsymbol{0} \prec \boldsymbol{B}^{-1} \preceq \boldsymbol{A}^{-1}$.

Exercise 4

Let $\boldsymbol{A}, \boldsymbol{B} \succ \boldsymbol{0}$ be positive definite matrices such that $\frac{1}{K}\boldsymbol{A} \preceq \boldsymbol{B} \preceq \boldsymbol{A}$ for some $K > 1$.

1. Prove that for $0 < \alpha \leq 1$,

$$\left\|\boldsymbol{I} - \alpha\boldsymbol{B}^{-1}\boldsymbol{A}\right\|_{\boldsymbol{A}\to\boldsymbol{A}} \leq \max(1-\alpha, |\alpha K - 1|)$$

Algorithm 1: CLIQUESAMPLE($v, \boldsymbol{S}$)

Input: Graph Laplacian $\boldsymbol{S} \in \mathbb{R}^{V \times V}$, of a graph with multi-edge weights $\boldsymbol{w}$, and vertex $v \in V$
Output: $\boldsymbol{Y}_v \in \mathbb{R}^{V \times V}$ sparse approximation of CLIQUE($v, \boldsymbol{S}$)
$\boldsymbol{Y}_v \leftarrow \boldsymbol{0}_{n \times n}$;
foreach *Multiedge* $e = (v, i)$ *from* v *to a neighbor* i **do**
  Randomly pick a multi-edge (v, j) with probability $\frac{\boldsymbol{w}(v,j)}{\boldsymbol{w}_v}$;
  If $i \neq j$, let $\boldsymbol{Y}_v \leftarrow \boldsymbol{Y}_v + \frac{\boldsymbol{w}(i,v)\boldsymbol{w}(j,v)}{\boldsymbol{w}(i,v)+\boldsymbol{w}(j,v)} \boldsymbol{b}_{i,j} \boldsymbol{b}_{i,j}^\top$;
return $\boldsymbol{Y}_v$;

Algorithm 2: Approximate Gaussian Elimination

Input: Graph Laplacian $\boldsymbol{L}$ of connected weighted graph G.
Output: Lower triangular[a] $\boldsymbol{\mathcal{L}}$.
Let $\boldsymbol{S}_0 = \boldsymbol{L}$ where $\boldsymbol{L}$ is the Laplacian of G with each original edge split into $K = c \log^2 n$ multi-edges with $1/K$ times original weight for some large enough constant c;
Generate a random permutation π on $[n]$;
for $i = 1$ *to* $i = n - 1$ **do**
  $\boldsymbol{l}_i = \frac{1}{\sqrt{\boldsymbol{S}_{i-1}(\pi(i),\pi(i))}} \boldsymbol{S}_{i-1}(:, \pi(i))$;
  $\boldsymbol{S}_i = \boldsymbol{S}_{i-1} - \text{STAR}(\pi(i), \boldsymbol{S}_{i-1}) + \text{CLIQUESAMPLE}(\pi(i), \boldsymbol{S}_{i-1})$
$\boldsymbol{l}_n = \boldsymbol{0}_{n \times 1}$;
return $\boldsymbol{\mathcal{L}} = \left[\boldsymbol{l}_1 \cdots \boldsymbol{l}_n\right]$ *and* π;

[a] $\boldsymbol{\mathcal{L}}$ is not actually lower triangular. However, if we let $\boldsymbol{P}_\pi$ be the permutation matrix corresponding to π, then $\boldsymbol{P}_\pi \boldsymbol{\mathcal{L}}$ is lower triangular. Knowing the ordering that achieves this is enough to let us implement forward and backward substitution for solving linear equations in $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{L}}^\top$.

Exercise 5

In the lecture today, we studied Approximate Gaussian Elimination. We considered the matrices $\boldsymbol{L}_i = \boldsymbol{S}_i + \sum_{j=1}^{i} \boldsymbol{l}_i \boldsymbol{l}_i^\top$.

1. Prove that the sequence of $\{\boldsymbol{L}_i\}$ form a martingale. Conclude that $\mathbb{E}\left[\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{L}}^\top\right] = \boldsymbol{L}$.

We let $\boldsymbol{L}_{i,e} = \boldsymbol{L} + \sum_{j \leq i} \sum_{f \leq e} \boldsymbol{Y}_{i,e} - \mathbb{E}\left[\boldsymbol{Y}_{i,e} \mid \text{all previous samples}\right]$. Note we order the multi-edges of vertex $\pi(i)$ by the order in which they are processed by CLIQUESAMPLE($\pi(i), \boldsymbol{S}_{i-1}$).

2. Prove that when $e_{\text{last}}(i)$ is the last sampled edge for vertex i, we have $\boldsymbol{L}_{i,e_{\text{last}}(i)} = \boldsymbol{L}_i$.

We also defined a "stopped" version of the martingale.

$$\tilde{\boldsymbol{L}}_i = \begin{cases} \boldsymbol{L}_i & \text{if for all } (j,e) < (i, e_{\text{last}}(i)) \text{ we have } \boldsymbol{L}_i \preceq 1.5\boldsymbol{L} \\ \boldsymbol{L}_{i^*,e^*} & \text{for } (i^*, e^*) \text{ being the least } (i,e) \text{ such that } \boldsymbol{L}_{i,e} \not\preceq 1.5\boldsymbol{L} \end{cases} \tag{1}$$

3. Prove that the sequence of $\left\{\tilde{\boldsymbol{L}}_i\right\}$ form a martingale.

4. Prove that $\tilde{\boldsymbol{L}}_i \preceq 2\boldsymbol{L}$.
5. Prove that $0.5\boldsymbol{L} \preceq \tilde{\boldsymbol{L}}_i \preceq 1.5\boldsymbol{L}$ implies $0.5\boldsymbol{L} \preceq \boldsymbol{L}_i \preceq 1.5\boldsymbol{L}$.

Exercise 6

In the lecture today, we used the following lemma.

Lemma. *Let $\boldsymbol{L}$ be the Laplacian of a connected graph. Let $\boldsymbol{S}$ be the Laplacian of another graph on the same vertex set*[1]*. If each multiedge e of* $\textsc{Star}(v, \boldsymbol{S})$ *has bounded norm in the following sense,*

$$\left\| \boldsymbol{L}^{+/2} \boldsymbol{w}_{\boldsymbol{S}}(e) \boldsymbol{b}_e \boldsymbol{b}_e^\top \boldsymbol{L}^{+/2} \right\| \leq R,$$

then each possible sampled multiedge e' of $\textsc{CliqueSample}(v, \boldsymbol{S})$ *also satisfies*

$$\left\| \boldsymbol{L}^{+/2} \boldsymbol{w}_{\text{new}}(e') \boldsymbol{b}_{e'} \boldsymbol{b}_{e'}^\top \boldsymbol{L}^{+/2} \right\| \leq R.$$

1. Give a proof of the Lemma. *Hint: use that effective resistance is a distance.*

Exercise 7

In the lecture today, we showed that the algorithm for Approximate Gaussian Elimination has expected running time $O(m \log^3 n)$ in a graph with m vertices and n edges.

Consider the following variant of the algorithm: pick the next vertex to eliminate randomly among vertices with degree at most two times the current average.

Argue that the algorithm still works with w.h.p. and now runs in time $O(m \log^3 n)$ deterministically.

[1] Think of $\boldsymbol{L}$ as the original Laplacian. When we use the lemma, $\boldsymbol{S}$ will be some intermediate Laplacian appearing during Approximate Gaussian Elimination.

Bonus Question

Another bonus question for those who want more. I probably won't discuss the solution in class.

Exercise 8

Let P_n be the path from vertex 1 to n and $G_{1,n}$ be the graph with only the edge between vertex 1 and n. Furthermore, assume that the edge between vertex i and $i+1$ has positive weight w_i for $1 \leq i \leq n-1$. Prove that

$$G_{1,n} \preceq \left(\sum_{i=1}^{n-1} \frac{1}{w_i} \right) \sum_{i=1}^{n-1} w_i G_{i,i+1}.$$