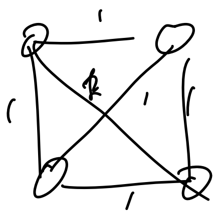
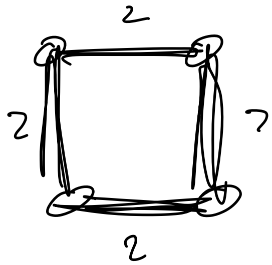



G




H

$$\forall x \quad x^T A x \le x^T B x$$

~~$\forall A$~~ $\quad A \preceq B$

---

$$n\{\overbrace{A}^{n}\,\underline{x} = b$$

$$n^{\omega} \qquad \omega \approx 2.37$$

$$A^{-1}$$

$$A\underline{x} = \underline{b}$$

$\downarrow$

$$nnz(A) = m \qquad mn$$

Peng Vempala

$m = 10n$ ~~PV~~ $n^{2.31}$

$n^{2+1/3}$

$$\forall \underline{x}.\ \underline{x}^T \tilde{L} \underline{x} \le (1+\varepsilon)\, \underline{x}^T L \underline{x}$$

$\Updownarrow$ ? <u>FALSE</u>

$$\cancel{\|\tilde{L}\| \le (1+\varepsilon)\|L\|}$$

$$\underline{x}^T(\tilde{L} - L)\underline{x} \le \varepsilon\, x^T L x \quad ?$$

$$\cancel{\|\tilde{L} - L\| \le \varepsilon}$$

$$\cancel{\underline{x}^T I (\tilde{L} - L)\underline{x} \le \varepsilon\, \underline{x}^T \underline{x}}$$

$$y = L^{1/2} \underline{x}$$

$$\underline{x} = L^{-1/2} \underline{y}$$

$$\forall y.\ y^T \underline{L^{-1/2}(\tilde{L} - L)L^{-1/2}}\, \underline{y} \le \varepsilon\, \underline{y}^T L^{-1/2} L L^{-1/2} \underline{y}$$

$$\| L^{-1/2}(\tilde{L} - L) L^{-1/2} \| \lesssim \varepsilon$$

Normalizing Map  Isotropic Position

$$\Phi(M) = L^{-1/2} M L^{-1/2}$$

Proving Spielman - Srivastava Spectral Sparsification

1) Use isotropic position
$\downarrow$ convert to spectral norm error

2) Calculate the expectation of $\tilde{L}$

3) Calculate sample norm $R$
and variance $\sigma^2$ ~~#~~

4) Apply Bernstein

5) Calculate # edges in $\tilde{E}$

$|\tilde{E}|$

---

$\tilde{E}$ $\quad L = \underline{B} W \underline{B}^T$

$$= \sum_e \underline{w}(e)\, \underline{b}_e \underline{b}_e^T$$

$$\tilde{L} = \sum_e \tilde{\underline{w}}(e)\, \underline{b}_e \underline{b}_e^T$$

$$\tilde{L} \approx_\varepsilon L$$

$$\Updownarrow$$

$$\| \Phi(\tilde{L}) - \Phi(L) \| \leq \varepsilon$$

$0 < \varepsilon < 1$

(2) Expectation

$$\mathbb{E}\tilde{L} = L$$

$$\mathbb{E}\tilde{L} = \mathbb{E}\sum_e \underline{\tilde{w}}(e)\, \underline{b}_e \underline{b}_e^T$$

$$= \sum_e \underbrace{\left(\mathbb{E}\,\underline{\tilde{w}}(e)\right)}_{\underline{w}(e)} \underline{b}_e \underline{b}_e^T$$

(3) $\underbrace{X_1, \dots, X_m}_{|E|}$

$$\Phi\left(\tilde{L} - L\right) = \Phi\left( \sum_e \underbrace{\left(\tilde{w}(e) - w(e)\right) \underline{b}_e \underline{b}_e^T}_{X_e} \right)$$

$$\|X_e\| \leq \left\| \frac{1}{p(e)} w(e) \Phi(b_e b_e^T) \right\|$$

$$\frac{1}{p(e)} \underbrace{\| w(e) \Phi(b_e b_e^T) \|}_{\text{leverage score } \ell(e)}$$

$\|$

$w(e)$ effective resistance across $e$

$$\underbrace{p(e) = \min(1, \alpha \ell(e))}$$

$$\|X_e\| \leq \frac{1}{\alpha}$$

$$w(e) \quad \underbrace{\| \Phi(b_e b_e^T) \|}$$

$$w(e) \, \| \underbrace{L^{-1/2} b_e}_{z} \underbrace{b_e^T L^{-1/2}}_{z^T} \|$$

$$w(e) \quad \| z z^T \| = z^T z$$

$$w(e) \cdot b_e^T L^{-1} b_e$$

VARIANCE : exercise

④ Apply Bernstein

$$\alpha = \frac{\log(n/\delta)}{\varepsilon^2}$$

success w. prob $1-\delta$

⑤ How many samples?

$$\mathbb{E}|\tilde{E}| = \sum_e p(e)$$

$$= \alpha \sum_e \ell(e)$$

$$= \alpha \sum_e \| w(e)\, \Phi(b_e b_e^T) \|$$

$$\alpha \sum_e \operatorname{tr}\left( w(e)\, \Phi(b_e b_e^T) \right)$$

$$= \alpha \operatorname{tr}\left( \underbrace{\Phi}_{} \left( \underbrace{\sum_e w(e)\, b_e b_e^T}_{L} \right) \right)$$

$$= \alpha \operatorname{tr}(I) = \alpha \cdot n$$

$$= \frac{\log(n/\delta)}{\varepsilon^2}\, n$$

Bernstein for scalars $(n = 1)$

$$\Pr[\, \|X\| \geq t \,]$$

$$\Pr[\, |X| \geq t \,]$$

$$= \Pr[\, X \geq t \text{ or } X \leq -t \,]$$

$$\Pr[\, X \geq t \,] \qquad \theta > 0$$

$$= \Pr[\, \exp(\theta X) \geq e^{\theta t} \,]$$

$$\leq \frac{\mathbb{E}\exp(\theta X)}{e^{\theta t}}$$

$$X = \sum_i X_i$$

$$\mathbb{E}\exp(\theta X)$$

$$= \prod_i \underbrace{\mathbb{E}\exp(\theta X_i)}$$

$$\exp(\theta X) =$$

$$= \exp\left(\theta \sum_i X_i\right)$$

$$= \prod_i \exp(\theta X_i)$$

FALSE FOR MATRICES

$$\mathbb{E}\exp(\theta X_i)$$

$$|X_i| \le R$$

$$\theta \le 1/R$$

$$|X_i \theta| \le 1$$

$$\mathbb{E}\exp(\_\,)$$

$$\mathbb{E}\exp(\theta X_i) \le \mathbb{E}\ 1 + \underbrace{\theta X_i}_{0} + \underline{\theta^2 X_i^2}$$

$$1 + \theta^2 \mathbb{E} X_i^2$$

$$1 + x \le e^x$$

$$\le e^{\theta^2 \mathbb{E} X_i^2}$$

$$\le \exp\left(\sum_i \theta^2 R\, \mathbb{E} X_i^2\right)$$

Choose $\theta$ $\left(0 < \theta \le 1/R\right)$

$$\Pr\left[\ \|X\| \geq t\ \right]$$

$$\Pr\left[\ \underbrace{\lambda_{\max}(X) \geq t}\ \text{or}\ \lambda_{\min}(X) \leq -t\ \right]$$

$$\Pr\left[\lambda_{\max}(X) \geq t\right]$$

$$\cancel{\leq} = \Pr\left[\lambda_{\max}(\exp(\theta X)) \geq e^{\theta t}\right]$$

$$\leq \Pr\left[\operatorname{tr}(\exp(\theta X)) \geq e^{\theta t}\right]$$

$$\mathbb{E}\ \operatorname{tr}(\exp(\theta X))$$

$$\mathbb{E}_{X_1} \cdots \mathbb{E}_{X_n} \ln\Big(\exp\big(\underbrace{\theta X_1 + \cdots}_{\#} + \theta X_n\big)\Big)$$

$$\frac{\log\big(\overbrace{\exp(\theta X_n)}^{A}\big)}{\log\big(\mathbb{E}\exp(\theta X_n)\big)}$$

$$\log\big(\mathbb{E}\exp(\theta X_i)\big)$$

$$\leq \theta^2 \mathbb{E} X_i^2$$