## Exercises

1. Let $G=(V, E)$ be a $d$-regular graph that is 3 -colorable and such that there is a 3 -coloring in which the color classes have equal size $|V| / 3$. Let $A$ be the adjacency matrix and $\frac{1}{d} \cdot M$ be the normalized adjacency matrix. Prove that $M$ has at least two eigenvalues which are smaller than or equal to $-1 / 2$, that is, $\lambda_{n-1} \leq-1 / 2$.
[Note: you get partial credit if you prove that there is a negative absolute constant, independent of $|V|$, such that two eigenvalues must be smaller than that constant.]

Give an example in which the bound the tight.
Show that the converse is not true. (That is, give an example of a regular graph that is not 3-colorable but such that at least two eigenvalues of the normalized adjacency matrix are $\leq-1 / 2$.)
2. Recall that, given two graphs $G=\left(V, E_{G}\right)$ and $H=\left(V, E_{H}\right)$, the non-uniform sparsest cut is

$$
\phi(G, H)=\min _{S \subseteq V} \frac{\frac{1}{\left|E_{G}\right|} \cdot \sum_{u, v} A_{u, v}\left|1_{S}(u)-1_{S}(v)\right|}{\frac{1}{\left|E_{H}\right|} \cdot \sum_{u, v} B_{u, v}\left|1_{S}(u)-1_{S}(v)\right|}
$$

where $A$ is the adjacency matrix of $G$ and $B$ is the adjacency matrix of $H$, and the minimum is taken over all sets $S$ that are not empty and are different from $V$.
Consider the following continuos relaxation

$$
\gamma(G, H)=\min _{x \in R^{V}} \frac{\frac{1}{\left|E_{G}\right|} \cdot \sum_{u, v} A_{u, v}|x(u)-x(v)|^{2}}{\frac{1}{\left|E_{H}\right|} \cdot \sum_{u, v} B_{u, v}|x(u)-x(v)|^{2}}
$$

Note that if $H$ is a clique with self-loops and $G$ is regular, then $\gamma(G, H)=$ $1-\lambda_{2}$ and $\phi(G, H)=\phi(G)$. Recall also that $\phi(G) \leq \sqrt{8\left(1-\lambda_{2}\right)}$, and so we may hope that, say, when $G$ and $H$ are two arbitrary regular graphs, we have $\phi(G, H) \leq O(\gamma(G, H)))$.
Give a counterexample by showing (an infinite family of) regular graphs $G, H$ such that $\phi(G, H) \geq \Omega(1)$ but $\gamma(G, H)=o(1)$.
[Notes: you get full credit even if $G$ and $H$ are not regular. You should be able to get a family of graphs for which $\gamma(G, H)=O(1 / n)$ and $\phi(G, H)=\Omega(1)$.]
[Hint: Let $G$ be a cycle]

