Exercises

1. Let \( G = (V, E) \) be a \( d \)-regular graph that is 3-colorable and such that there is a 3-coloring in which the color classes have equal size \( |V|/3 \). Let \( A \) be the adjacency matrix and \( \frac{1}{d} \cdot M \) be the normalized adjacency matrix. Prove that \( M \) has at least two eigenvalues which are smaller than or equal to \(-1/2\), that is, \( \lambda_{n-1} \leq -1/2 \).

[Note: you get partial credit if you prove that there is a negative absolute constant, independent of \( |V| \), such that two eigenvalues must be smaller than that constant.]

Give an example in which the bound the tight.

Show that the converse is not true. (That is, give an example of a regular graph that is not 3-colorable but such that at least two eigenvalues of the normalized adjacency matrix are \( \leq -1/2 \).)

2. Recall that, given two graphs \( G = (V, E_G) \) and \( H = (V, E_H) \), the non-uniform sparsest cut is

\[
\phi(G, H) = \min_{S \subseteq V} \frac{1}{|E_G|} \cdot \sum_{u,v} A_{u,v} |1_S(u) - 1_S(v)|
\]

where \( A \) is the adjacency matrix of \( G \) and \( B \) is the adjacency matrix of \( H \), and the minimum is taken over all sets \( S \) that are not empty and are different from \( V \).

Consider the following continuos relaxation

\[
\gamma(G, H) = \min_{x \in \mathbb{R}^V} \frac{1}{|E_G|} \cdot \sum_{u,v} A_{u,v} |x(u) - x(v)|^2
\]

Note that if \( H \) is a clique with self-loops and \( G \) is regular, then \( \gamma(G, H) = 1 - \lambda_2 \) and \( \phi(G, H) = \phi(G) \). Recall also that \( \phi(G) \leq \sqrt{8(1 - \lambda_2)} \), and so we may hope that, say, when \( G \) and \( H \) are two arbitrary regular graphs, we have \( \phi(G, H) \leq O(\gamma(G, H)) \).

Give a counterexample by showing (an infinite family of) regular graphs \( G, H \) such that \( \phi(G, H) \geq \Omega(1) \) but \( \gamma(G, H) = o(1) \).

[Notes: you get full credit even if \( G \) and \( H \) are not regular. You should be able to get a family of graphs for which \( \gamma(G, H) = O(1/n) \) and \( \phi(G, H) = \Omega(1) \).

[Hint: Let \( G \) be a cycle]