Exercise 1 (Consensus) Suppose there are two players each of which has access to a real number. In particular, let $a_1 \in \mathbb{R}$ be a number that is available to player 1 but not visible to player 2 and $a_2 \in \mathbb{R}$ be a number that is available to player 2 but not visible to player 1. Show that there does not exist a *determinsitic consensus procedure*, i.e., a function $\ell : \mathbb{R} \to \mathbb{R}$ satisfying

$$\forall a_1, a_2 \in \mathbb{R}, |a_1 - a_2| \leq 1 : \ell(a_1) = \ell(a_2) \leq \min\{a_1, a_2\}\,.$$

Exercise 2 (Truthfulness) Prove the following (so-called direct) characterization of truthfulness.

A mechanism for multi-unit auctions with n bidders and m items is truthful if and only if it satisfies the following two properties for every i and every v_{-i}:

i) For every number $k \in \{0, \ldots, m\}$ of items, there exists a price $q_k^{(i)}(v_{-i})$. That is, for all v_i with $f_i(v_i, v_{-i}) = k$, it holds $p_i(v_i, v_{-i}) = q_k^{(i)}(v_{-i})$, where f denotes the social choice function and p the payments.

ii) The social choice function maximizes the utility for player i. That is,

$$f(v) = \operatorname*{argmax}_{(s_1,\ldots,s_n)\in A^{(i)}(v_{-i})} (v_i(s_i) - q_{s_i}^{(i)}(v_{-i}))$$

with $A^{(i)}(v_{-i}) \subseteq \{0, \ldots, m\}^n$ being any non-empty subset of feasible allocations.

Exercise 3 (Pareto-optimal solutions) Suppose we are given n objects each coming with a profit $p_i > 0$ and a weight $w_i > 0$. A subset $S \subseteq \{1, \ldots, n\}$ with weight $w(S) = \sum_{i\in S} w_i$ and profit $p(S) = \sum_{i\in S} p_i$ *dominates* another subset $T \subseteq [n]$ if $w(S) \leq w(T)$ and $p(S) \geq p(T)$. A set that is not dominated by any other set with smaller weight or larger profit is called *Pareto-optimal.* The set of Pareto-optimal solutions is called the *Pareto front.*

Present an algorithm that generates the Pareto-front in an output efficient way, that is, in time $O(\sum_{i=1}^{n-1} q_i)$, where q_i denotes the size of the Pareto front over the objects $1, \ldots, i$.