The quiz is anonymous. Your solution will be identified by the number printed in the header and the top right corner of the first page. Please tear off the corner and take it with you. There are 20 problems. Please return your completed sheet at the end of the lecture.

Good luck!

Analysis

Problem 1. What do the following functions converge to?

1. $f(x) = x/(1-x)$ for $x \rightarrow \inf$

   Answer: -1

2. $f(x) = x^n/x(n+1)$ for $n > 1$ and for $x \rightarrow \inf$

   Answer: $+\inf$

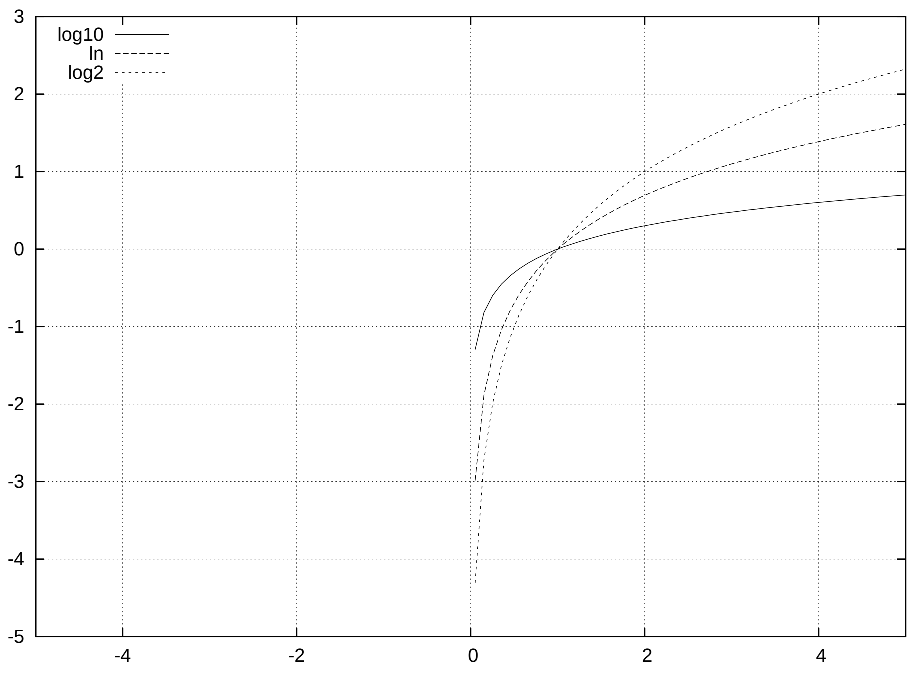
3. $f(k) = \sum_{i=1}^{k}(1/2)^k$ for $k \rightarrow \inf$

   Answer: 1

Logarithm

Problem 2. Please sketch the shape of a logarithmic function in the area below. You may ignore the basis of the logarithm, but assume it is larger than 1.

Answer:



Problem 3. Let n be a positive integer. Give, as a function of n, the least number of bits you need to represent n in binary code.

Answer: $\lfloor \log_2 n \rfloor + 1$

Problem 4. Heracles, the mighty hero of Greek mythology, attacks Hydra. Initially Hydra has only one head but every time Heracles cuts one of Hydra's heads, she grows two new heads. Heracles must cut all the heads of Hydra at once, or Hydra will eat him. But mighty as Heracles might be, he can only cut 130 heads at once.

1. How many times can Heracles cut all the heads of Hydra until she will have too many heads for Heracles to cut at once?

   Answer: 7 times. After that Hydra will have 256 heads.

2. Assume Heracles can cut n ($n \geq 2$) heads of Hydra at once. Give, as a function of n, the number of times Heracles can cut all the heads of Hydra at once (assuming Hydra starts with one head).

   Answer: $\lfloor \log_2 n \rfloor$ times. After that Hydra will have more than n heads.

Linear Algebra

Problem 5. What is the inner product (dot product) of the vectors (2 3 5) and (7 11 13)?

Answer: Their inner product is $2 \cdot 7 + 3 \cdot 11 + 5 \cdot 13 = 112$.

Problem 6. Are the above vectors orthogonal? Why?

Answer: No, their inner product is not zero.

Problem 7. Consider vectors (2 3 5) and (7 11 13). Are these two vectors linearly independent? Why?

Answer: Yes. It is enough to show that there exists no such x that $x \cdot (2\ 3\ 5) = (7\ 11\ 13)$. And $x \cdot 2 = 7$ means that $x = 3.5$, but $3.5 \cdot 3 = 10.5! = 11$ and therefore no such x can exist.

Problem 8. What is the rank of the following matrix?

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 1.5 & 2.5 \\ 7 & 11 & 13 \\ 9 & 14 & 18 \end{pmatrix}$$

Answer: Rank is two. Second row vector is 1/2 times the first and fourth is first plus third. First and third row vectors are linearly independent, see above.

Problem 9. Write the transpose of the matrix A above.

Answer:

$$A^T = \begin{pmatrix} 2 & 1 & 7 & 9 \\ 3 & 1.5 & 11 & 14 \\ 5 & 2.5 & 13 & 18 \end{pmatrix}$$

Problem 10. If an n-by-n matrix has rank r, what is its nullity (i.e. the dimensionality of its null space)?

Answer: Per the rank-nullity theorem, $nullity = n - rank$.

Combinatorics

Problem 11. What is the difference between a *set* and a *bag*?

Answer: Set is disjoint, bag not.

Problem 12. How many different subsets does a set of size n have?

Answer: 2^n.

Problem 13. How many different subsets of size k does a set of size n have?

Answer:

$$\binom{n}{k} = \frac{n!}{(k!(n-k)!)}.$$

Probabilities

Problem 14. Describe (in your own words) what the difference between *probability theory* and *statistics* is.

Answer: Probability theory describes the properties of the outcome of an experiment or (more generally) the outcome of a data-generating process. Statistics lets us draw conclusions about the data-generating process, given that we can observe the outcome.

Problem 15. Describe (in your own words) what a *random variable* is.

Answer: A random variable X maps a set of events to a value x. If the events follow the axioms of probability theory, then we can calculate a probability for every value x of X.

Problem 16. Let A, B, and C be three non-independent events with

$$\begin{aligned} \Pr(A) &= 0.7, & \Pr(B) &= 0.5, \\ \Pr(C) &= 0.4, & \Pr(A \wedge B) &= 0.4, \\ \Pr(A \wedge C) &= 0.3, & \Pr(B \wedge C) &= 0.2, \\ \Pr(A \wedge B \wedge C) &= 0.1. \end{aligned}$$

What is the probability that either A, B, or C happens (i.e. $\Pr(A \vee B \vee C)$)?

Answer: By inclusion-exclusion formula, $\Pr(A \vee B \vee C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \wedge B) - \Pr(A \wedge C) - \Pr(B \wedge C) + \Pr(A \wedge B \wedge C) = 0.7 + 0.5 + 0.4 - 0.4 - 0.3 - 0.2 + 0.1 = 0.8$.

Problem 17. Let X_1 and X_2 be two independent binary random variables with $\Pr(X_1 = 1) = 0.8$ and $\Pr(X_2 = 1) = 0.5$. Let Y be their sum, $Y = X_1 + X_2$.

1. What is $\mathbf{E}(X_1)$?

   Answer: $\mathbf{E}(X_1) = 0.8$

2. What is $\mathbf{E}(Y)$?

   Answer: $\mathbf{E}(Y = \mathbf{E}(X_1 + X_2) = \mathbf{E}(X_1) + \mathbf{E}(X_2) = 0.8 + 0.5 = 1.3$ N.B. the linearity of expectation.

3. What is $\mathbf{E}(Y \mid X_2 = 0)$?

   Answer:

$$\begin{aligned} \mathbf{E}(Y \mid X_2 = 0) &= \sum_{y=0}^{2} y \cdot \Pr(Y = y \mid X_2 = 0) = 0 \cdot \Pr(Y = 0 \mid X_2 = 0) \\ &\quad + 1 \cdot \Pr(Y = 1 \mid X_2 = 0) + 2 \cdot \Pr(Y = 2 \mid X_2 = 0) \\ &= \Pr(Y = 1 \mid X_2 = 0) = \Pr(X_1 = 1) = 0.8 \end{aligned}$$

Problem 18. Let X be Binomially distributed random variable with parameters n and p, $X \sim Binom(n, p)$. What is the variance of X?

Answer: $np(1 - p)$.

Miscellaneous

Problem 19. Given a list of n numbers, explain what is the difference between the *mean* and the *median* of those numbers?

Answer: Let the numbers be $x_1 \ldots, x_n$. The mean (or the arithmetic average) is $\frac{1}{n}\sum_n x_x$. The median m is such a value the half of the set is less than m and half is greater than m. If we arrange the x_i in the ascending order, the median will be the element at position $\lceil n/2 \rceil$.

Problem 20.

How much wood would a woodchuck chuck
if a woodchuck could chuck wood?

Answer:

He would chuck, he would, as much as he could,
and chuck as much as a woodchuck would
if a woodchuck could chuck wood.