Symmetry in Shapes – Theory and Practice

Niloy Mitra *Maksim Ovsjanikov* *Mark Pauly* *Michael Wand* *Duygu Ceylan*

UCL
ÉCOLE POLYTECHNIQUE ParisTech
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
max planck institut informatik
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Introduction
Symmetry & Group Theory

Michael Wand

Saarland University / MPI Informatik

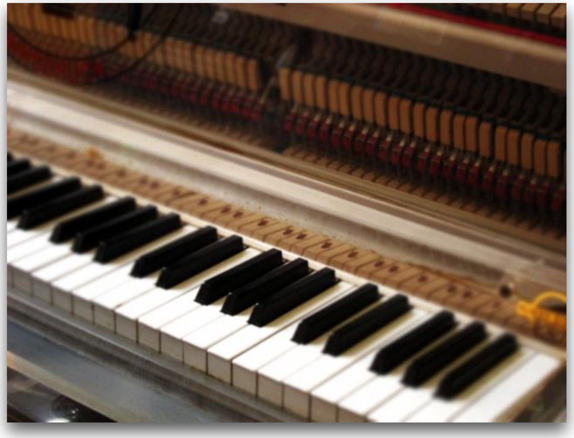
What is Symmetry?

Definition [Wikipedia]

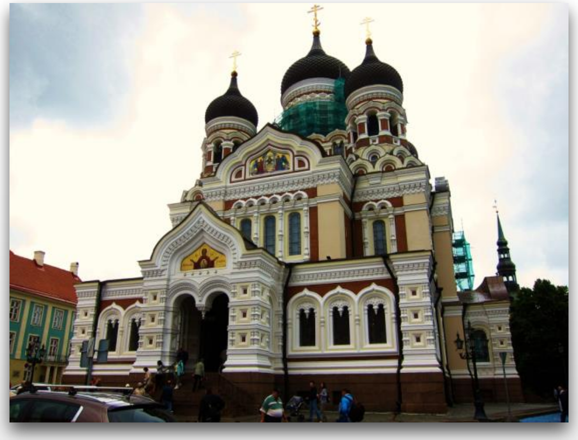
- "...a vague sense of harmonious and beautiful proportion and balance..."
- "...an exact mathematical 'patterned self-similarity' that can be demonstrated with the rules of a formal system..."

The second is a formalization of the first.





Group Theory

What is symmetry?

- The group of operations does not change the object

- Invariance of geometry under sets of transformations

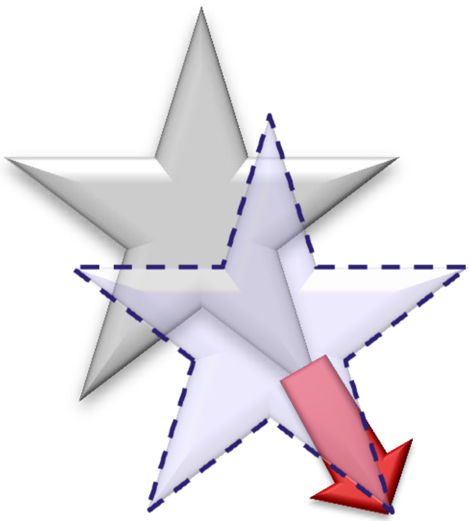
Geometric Symmetry

Invariance under transformations:

- Set of transformations
- Leaves object unchanged

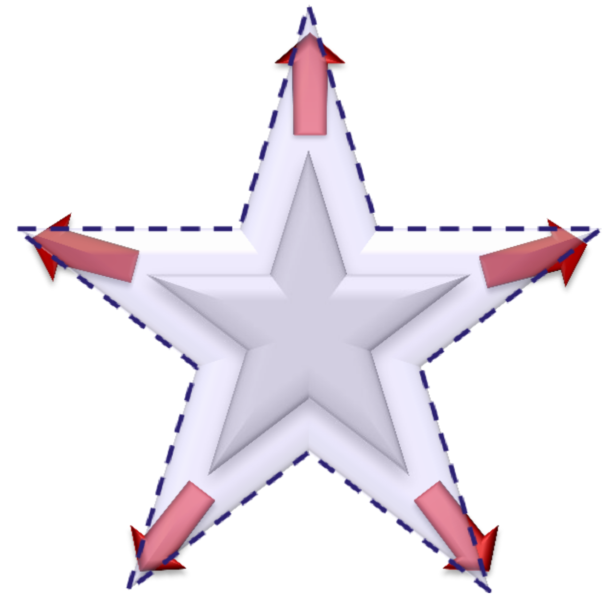
Transformations

Rigid (Similarity) Transformations



Translation

Rotation



(Scaling)

Geometric Symmetry

Invariance under transformations:

- Set of transformations
- Leaves object unchanged

Invariance to Transformations

Rotations that do not change the star:

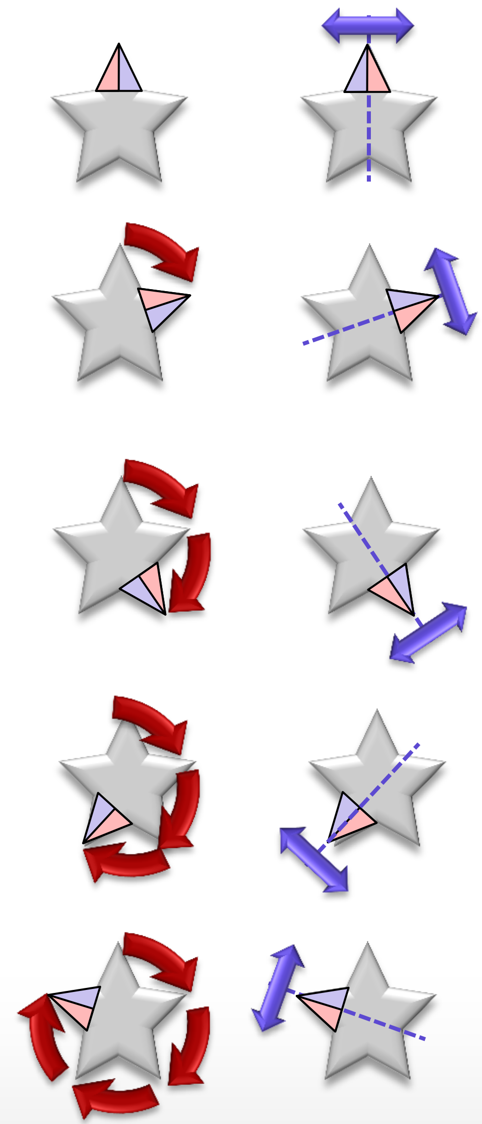
- $\frac{360°}{5} \cdot 0 = 0°$
- $\frac{360°}{5} \cdot 1 = 72°$
- $\frac{360°}{5} \cdot 2 = 144°$
- $\frac{360°}{5} \cdot 3 = 216°$
- $\frac{360°}{5} \cdot 4 = 288°$
- $\frac{360°}{5} \cdot 5 = 360° = 0°$

"Cyclic Group C_5"

Invariance to Transformations

Still no changes:

- $\frac{360°}{5} \cdot 0 = 0°$ $+$ *Reflection*
- $\frac{360°}{5} \cdot 1 = 72°$ $+$ *Reflection*
- $\frac{360°}{5} \cdot 2 = 144°$ $+$ *Reflection*
- $\frac{360°}{5} \cdot 3 = 216°$ $+$ *Reflection*
- $\frac{360°}{5} \cdot 4 = 288°$ $+$ *Reflection*
- $\frac{360°}{5} \cdot 5 = 360° = 0°$ $+$ *Reflection*

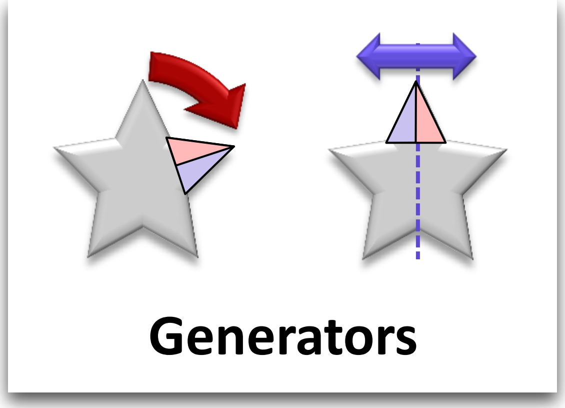


"Dihedral Group D_5"

Invariance to Transformations

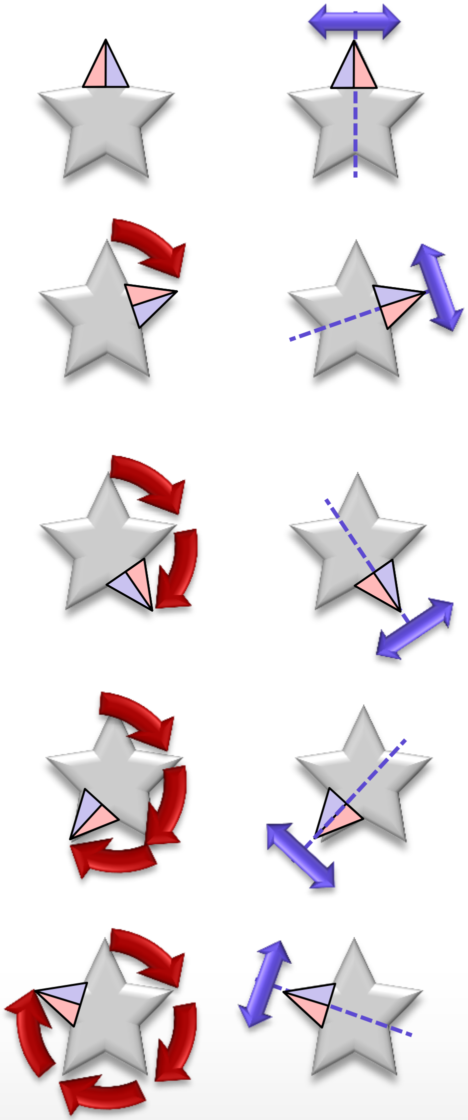
Generators

- For example:
- $\frac{360°}{5} \cdot 1 = 72°$
- *Reflection*



Combinations

- All combinations of the two generate the whole group
- Repetitions allowed



Formalization

Properties

- Closure: $a, b \in G$

  $\Rightarrow a \circ b \in G$

- Associativity: $\forall a, b, c \in \mathrm{G}$:

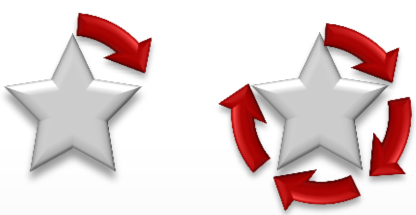
  $(a \circ b) \circ c = a \circ (b \circ c)$

- Identity: $\exists id \in G: \forall a \in \mathrm{G}$:

  $id \circ a = a \circ id = a$

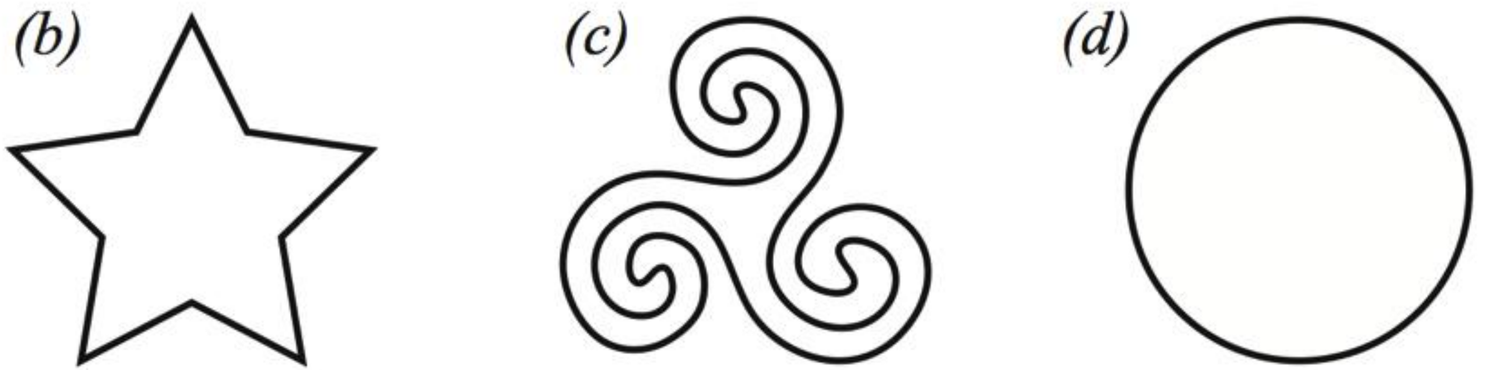
- Inverse: $\forall a \in \mathrm{G}: \exists a^{-1} \in G$:

  $a, b \in G \Rightarrow a \circ b \in G$



"Group Axioms"

More Examples...



dihedral group D_5 *cyclic group* C_3 *infinite group* $O(2)$

Formalization

Transformations of Space:

- Bijections $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ (usually: $d = 2,3$)
  - Transformations (permutations) of space
  - Form a group
- Rigid mappings, similarity transforms
  - Subgroups thereof
- Other choices
  - Translations, Reflections
  - Conformal mappings
  - Isometries
  - ...

Rigid Symmetry Groups

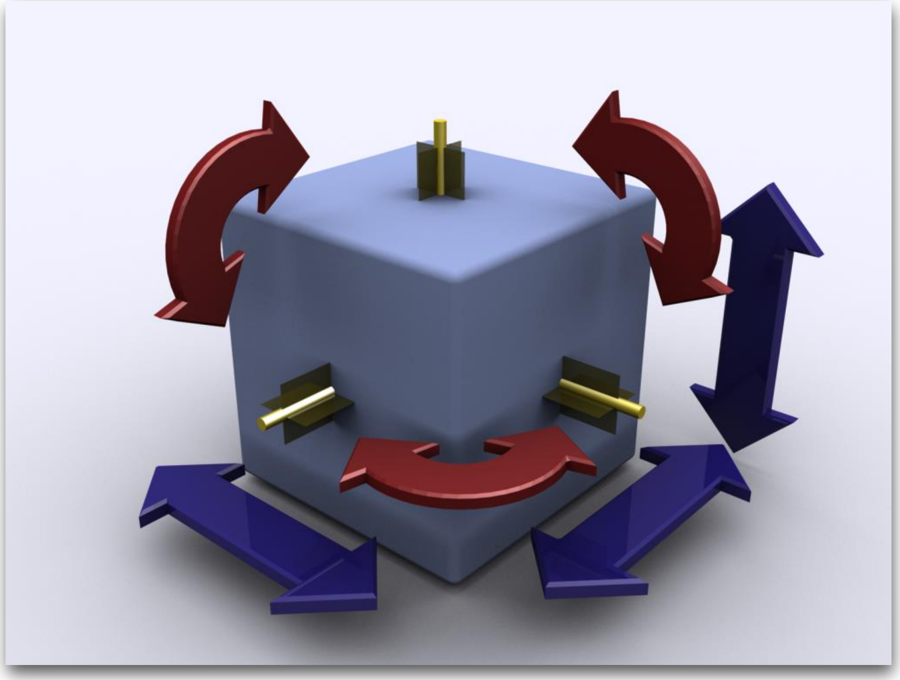
Classification

- Subgroups of the rigid mappings E(3)

Types

- Continuous vs. discrete
- Point-symmetries vs. lattices

Point Symmetries

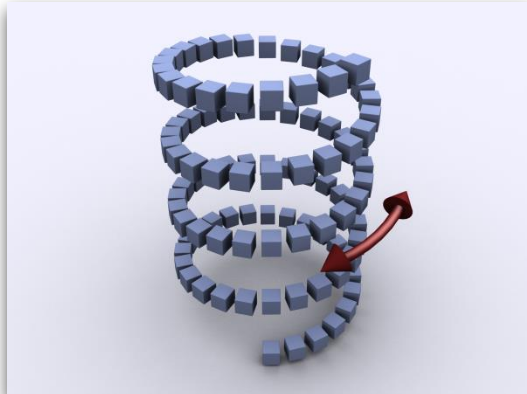


symmetries of a cube – O_h

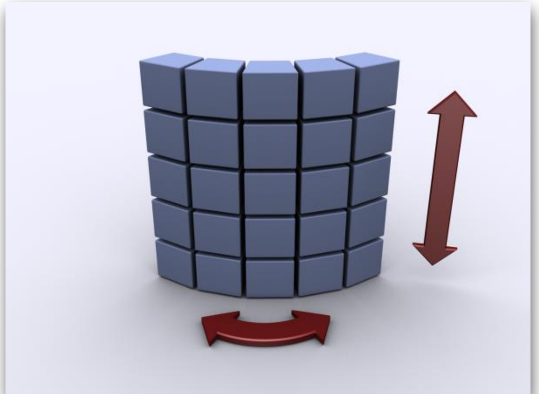
Point Symmetries

- Full classification known
- Full list e.g. on Wikipedia

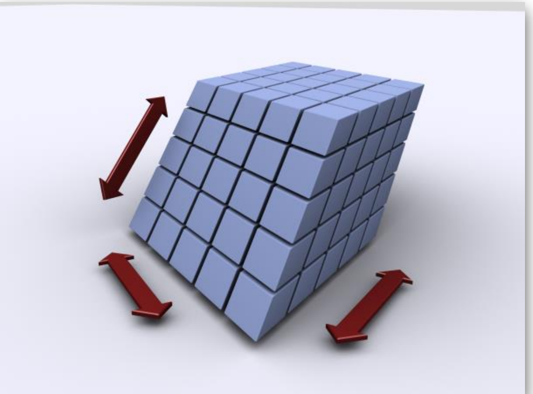
Lattices: k-Parameter Grids



1-parameter



2-parameter



3-parameter

k-Parameter Grids

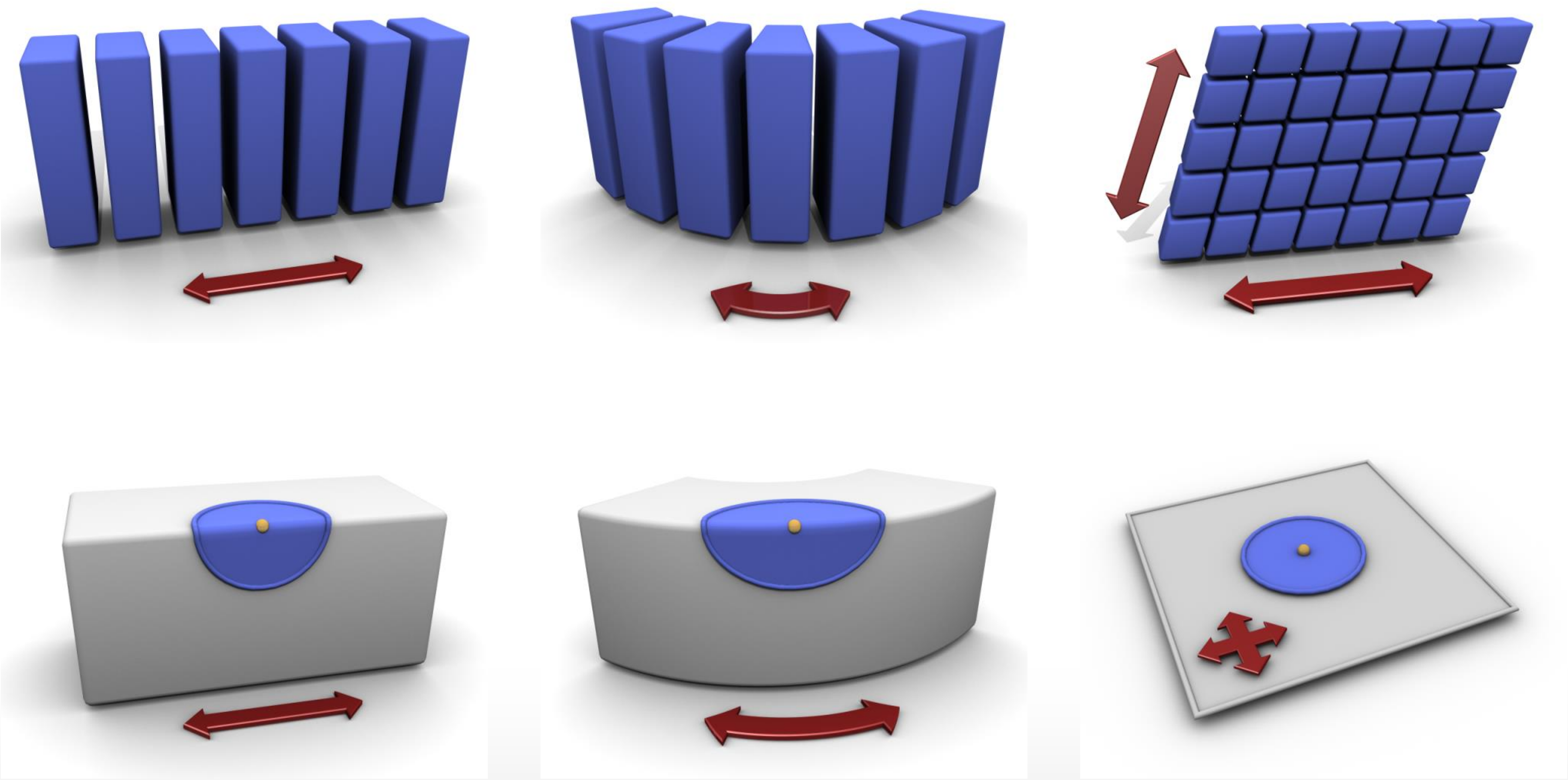
- Commutative generators $a, b, c \in E(3)$
- Can be expressed as

$$G = \{a^i \circ b^j \circ c^k | i, j, k \in \mathbb{Z}\}$$

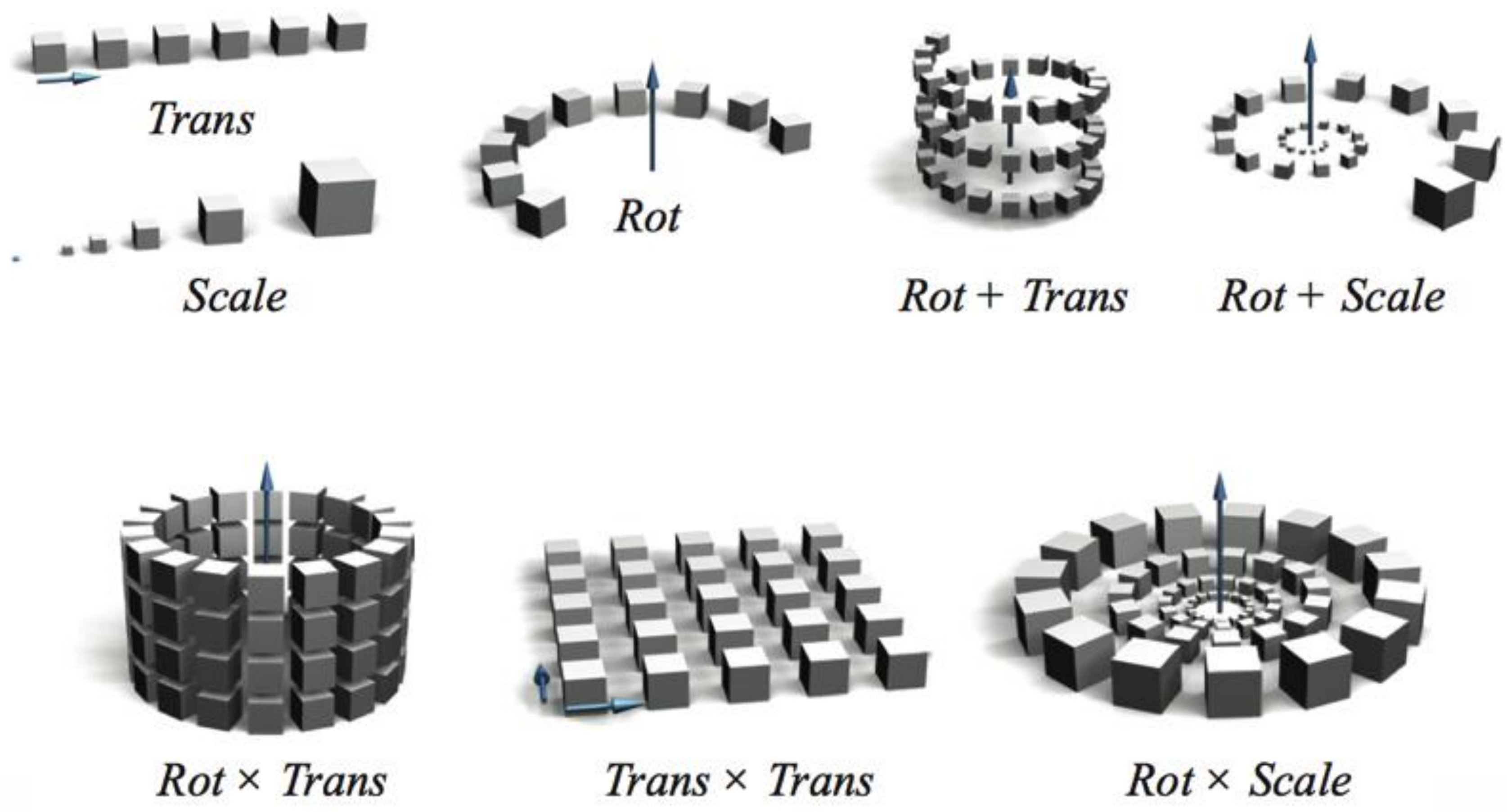
- Isomorphic to integer lattices (grids)

Lattices

Commutative Transformation Groups (Grids)

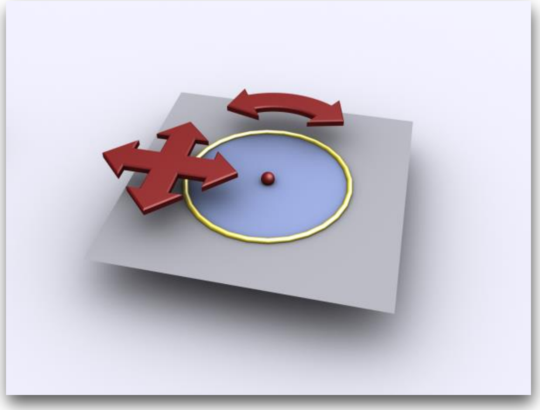


Including Scaling

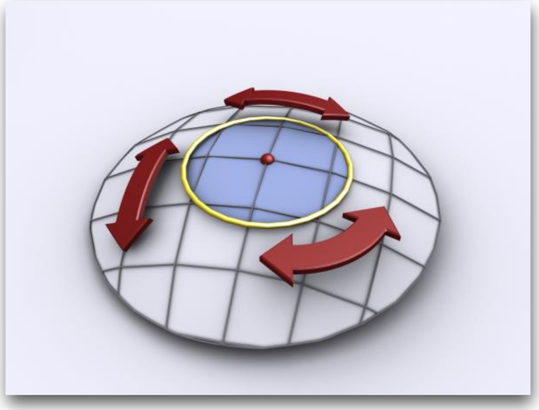


Continuous Symmetries

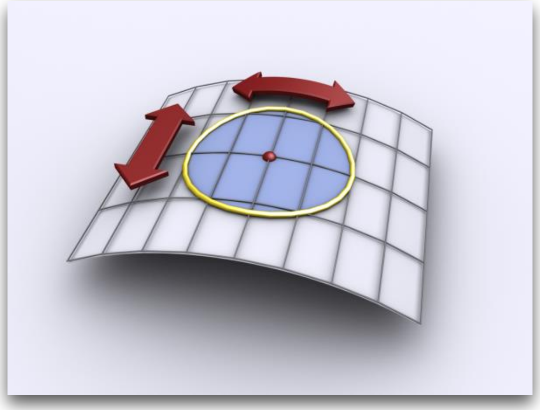
Slippable Surfaces (rigid motions)



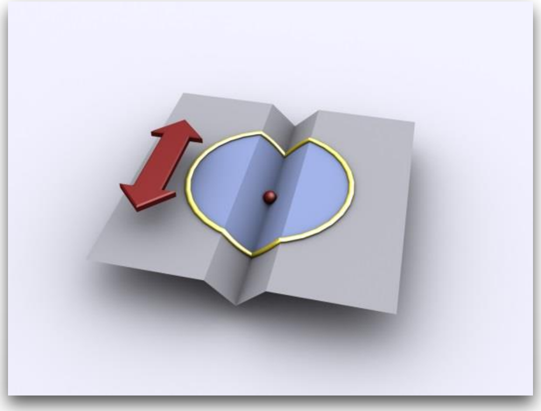
plane



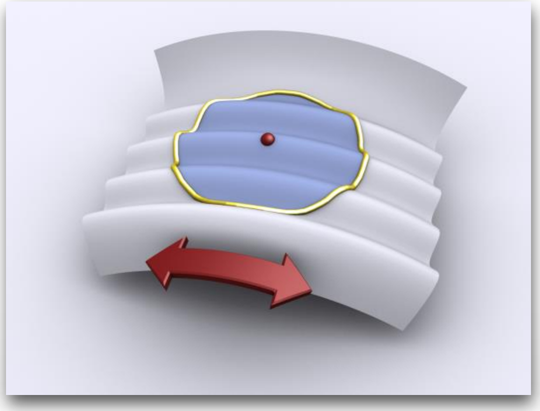
sphere



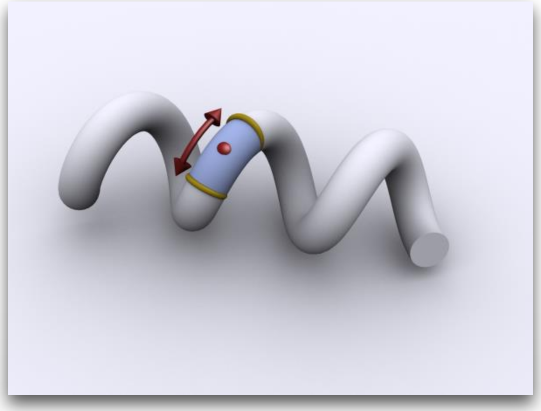
cylinder



linear extrusion



surface of revolution



helix

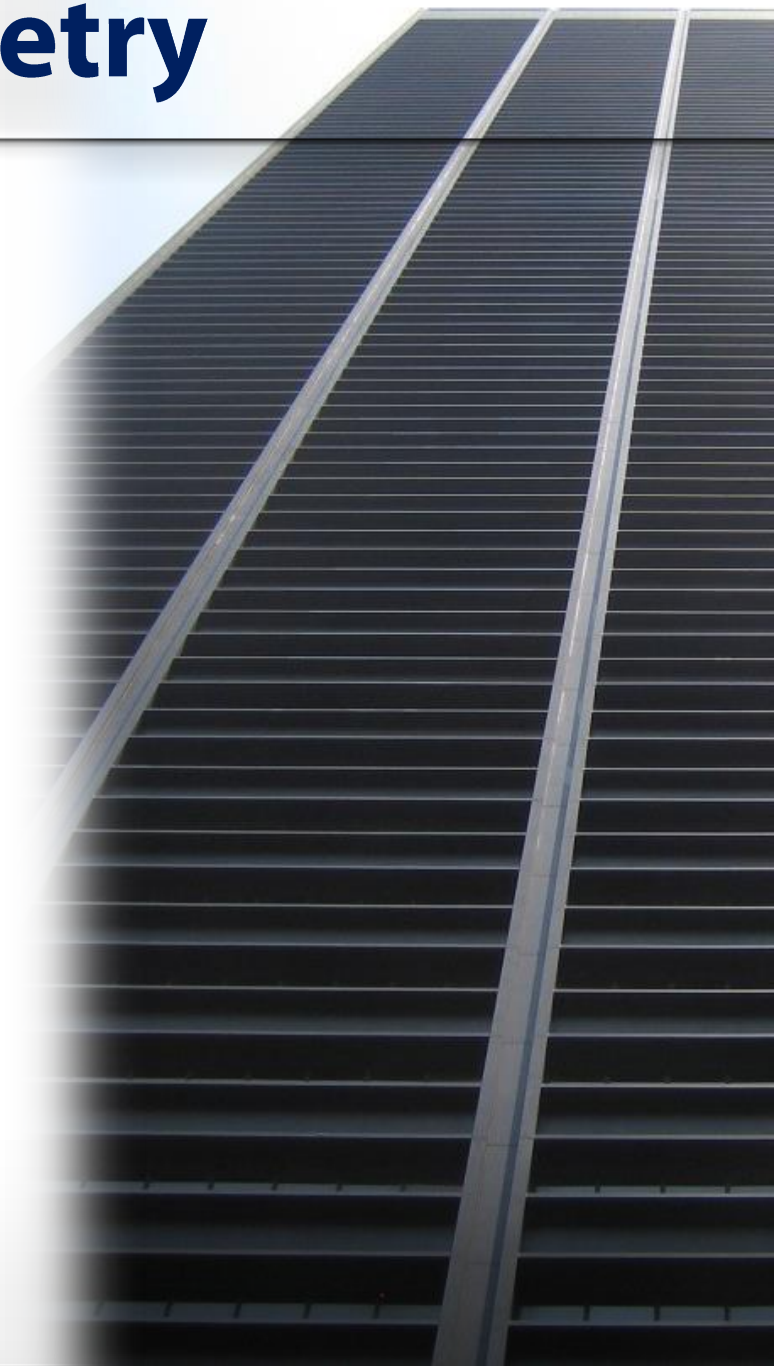
Mathematical Symmetry

What is symmetry?

- Invariance of geometry under *group actions*

- Looking at the object:

  Symmetry describes the absence of Information



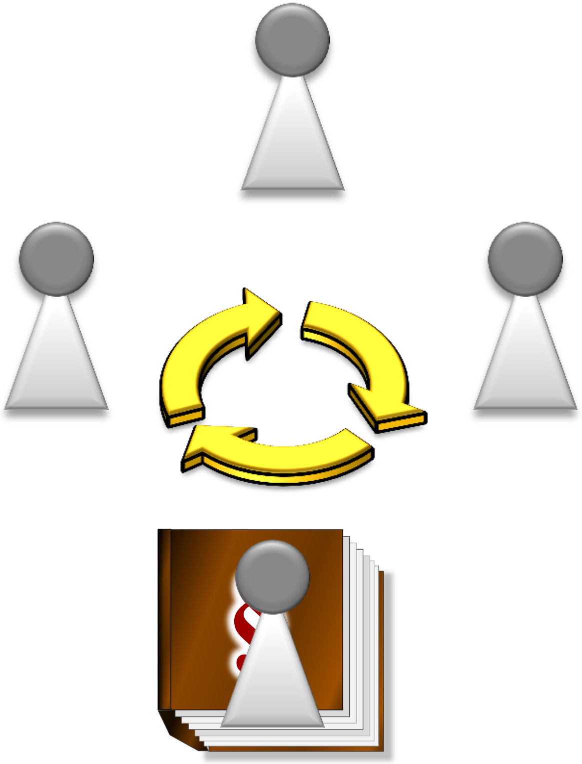
Influences beyond Math...

Democracy

- Citizens are equal under the law

Communism

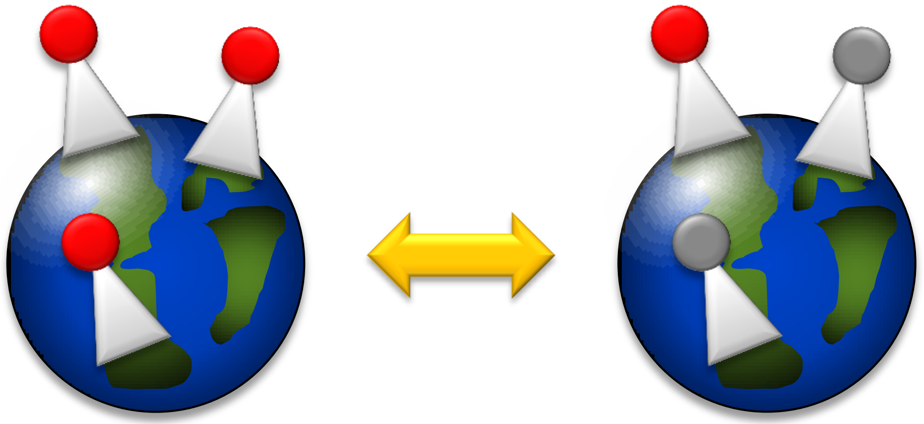
- Citizens are equal



Examples

Idea:

- Make theory invariant under the unknown



Philosophy

- Observer problem
- Everyday reasoning is invariant
  - Example: Turing test

Symmetry in Physics

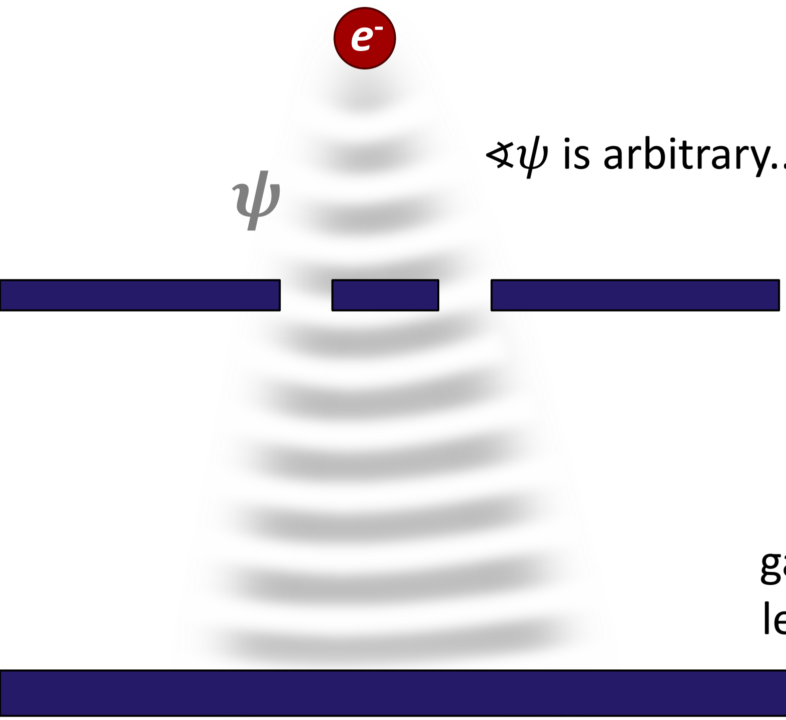
Physics: *symmetric laws*

- ***Newtonian mechanics***
  Invariant under
  Galilean transformations
- ***Maxwell equations / Special relativity***
  Lorentz invariance
- ***Quantum field theory***
  Gauge transformations/fields

$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

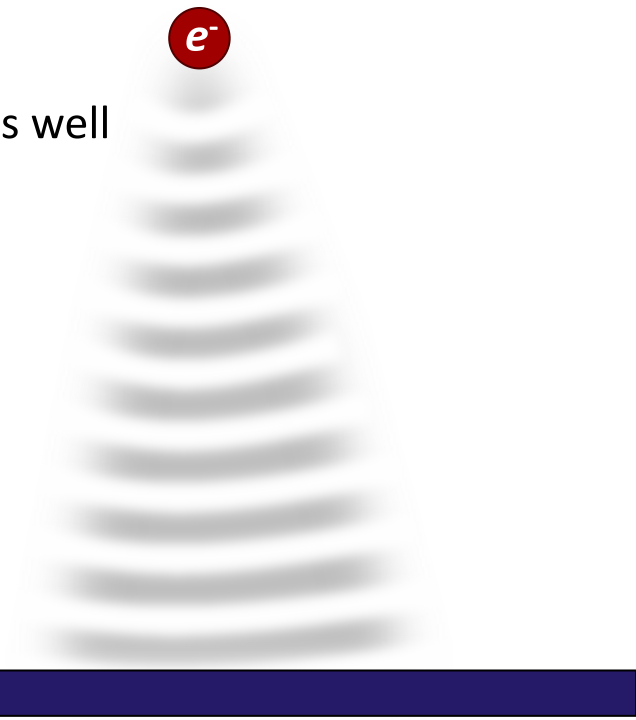
Maxwell's equations
[Wikipedia]

Quantum Mechanics



$\sphericalangle\psi$ is arbitrary...

...here as well



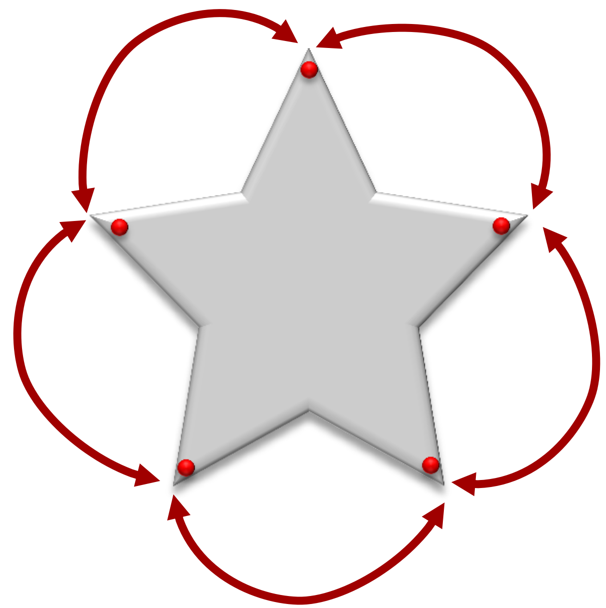
general case:
consistency?

solution:
gauge field: photons
leads to interactions

Summary of the Philosophy

Symmetry

- Some things we *don't/can't know*
  or *don't care about*
- Or the *information does not exist*



What to do

- Factor it out
  - $(Representation) \bmod (Symmetry)$
- Graphics: Redundancy, Invariants, Modeling