Homework 10

N. Megow, K. Mehlhorn
J. Mestre

This assignment is **due on July 1/2** in your tutorial session. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, however, must be *written individually* without consulting someone else's solutions or any other source like the web.

Problem 1 **(4 points)**
Let k be a positive integer and let $P := conv.hull(\{(0,0),(0,1),(k,\frac{1}{2})\})$. Show that the Chvátal rank of P is at least k. *Hint: Show that* $(k-1,\frac{1}{2}) \in P'$.

Problem 2 **(1+3+4 points)**
Consider the following assignment problem with a budget constraint: There is a set of n jobs to be assigned to a set of n workers, one job to one worker. Let c_{ij} be the profit gained when assigning worker i to job j, and let t_{ij} be the cost of training worker i to do job j. We are given a total training budget of b units. We want to find a feasible assignment of maximum profit subject to the budget constraint, that is,

$$\max \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1,\ldots,n \tag{1}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1,\ldots,n \tag{2}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}x_{ij} \le b \tag{3}$$

$$x_{ij} \in \mathbb{Z} \qquad i,j = 1,\ldots,n \tag{4}$$

(a) What can you say about the complexity status of this problem?

(b) Formulate the three following Lagrangean relaxations in which

1. (3) is dualized
2. (1) and (2) are dualized
3. (2) and (3) are dualized.

(c) Discuss for each of the relaxations: the ease of solution of the Lagrangian subproblems (interpretation, complexity), the ease of solution of the Lagrangian dual, and the strength of the resulting Lagrangian dual bound.

Problem 3 **(4+2 points)**
Consider the *knapsack problem*: Given is a weight bound K and a set of n items, each with a nonnegative weight w_j and *nonnegative* profit c_j, $j = 1,\ldots,n$. Determine a subset of items $S \subseteq \{1,\ldots,n\}$ of maximum total profit, $\sum_{j\in S} c_j$, such that $\sum_{j\in S} w_j \le K$.

Consider an integer programming formulation (ILP) where each item has associated a binary decision variable (see previous homework).

(a) Show that the following combinatorial algorithm gives an optimal solution to the LP-relaxation of (ILP); e.g, using complementary slackness conditions.

Algorithm: Order and reindex items such that

$$\frac{c_1}{w_1} \geq \frac{c_2}{w_2} \geq \cdots \geq \frac{c_n}{w_n}.$$

Let $k := \min\{j \in \{1, \ldots, n\} \mid \sum_{\ell=1}^{j} w_\ell > K\}$, and define

$$\begin{aligned} x_j &= 1, \qquad j = 1, \ldots, k-1 \\ x_k &= \frac{K - \sum_{j=1}^{k-1} w_j}{w_k} \\ x_j &= 0, \qquad j = k+1, \ldots, n\,. \end{aligned}$$

(b) Given the LP-solution from (a), consider the integral solution $S = \{1, 2, \ldots, k-1\}$. This corresponds to the algorithm that chooses S to be the maximal set of items that fits into the knapsack when adding items one after the other in non-increasing order of their ratios c_j/w_j. Does this algorithm give a good approximation? How large can the ratio between this solution and an optimal integral solution be in the worst case?