Homework 8

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This assignment is **due on June 17/18** in your tutorial session. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, however, must be *written individually* without consulting someone else's solutions or any other source like the web.

Please read carefully the following (corrected) definition.

Definition 1 *A matrix* $A \in \mathbb{Z}^{m \times n}$ *is* totally unimodular *if each square submatrix of A has determinant* $-1, 0,$ *or* 1. *A square* submatrix $B \in \mathbb{Z}^{k \times k}$ *of A is a square matrix that is obtained by deleting* $m - k$ *rows and* $n - k$ *columns of A.*

Problem 1 **(4 points)**
Show that for proving that a matrix A is totally unimodular, it is not enough to consider just submatrices involving consecutive rows and columns of A. How many squared submatrices has $A \in \mathbb{Z}^{m \times n}$?

Problem 2 **(4 points)**
Prove the following theorem using totally unimodularity.

Theorem 1 (The König-Egerváry Theorem (1931)) *In any bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.*

Problem 3 **(4 points)**
A matrix $A \in \{0,1\}^{m \times n}$ has the *consecutive* 1*'s property* (along columns), if in every column the 1's appear consecutively (assuming some linear ordering of rows of A). Such matrices are called *interval matrices.*
Show that any matrix with the consecutive 1's property is totally unimodular

Problem 4 **(4 points)**
Suppose we have n activities to choose from. Activity i starts at time s_i and ends at time t_i (or more precisely just before t_i); if chosen, activity i gives us a profit of p_i units. Our goal is to choose a subset of the activities which do not overlap (nevertheless, we can choose an activity that ends at t and one that starts at the same time t) and such that the total profit, that is, the sum of profits of the selected activities, is maximized.

1. Give an integer programming formulation of the form $\max\{p^T x \mid Ax \leq b, x \in \{0,1\}\}$ for this problem.
2. Show that the matrix A is totally unimodular, implying that one can solve this problem by solving the linear program $\{\max p^T x \mid Ax \leq b, 0 \leq x_i \leq 1 \text{ for every } i\}$. (Use the result you proved in Problem 3.)