Integer Linear Programming

Introduction

Given: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$

Task: Find $x \in \mathbb{Z}^n$, s.t. $Ax \le b$ and $c^T x$ is maximized.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \le b \qquad \text{(ILP)} \\ & x \in \mathbb{Z}^n \end{aligned}$$

More general: Mixed integer lin. programming (MILP); only some variables must be integral.

Powerful modelling framework:

- ▷ binary choice: $x_j \in \{0,1\}$ (knapsack, shortest path)
- ▷ relation between vars.: $\sum_{j=1}^{n} x_j \le 1$ $\Leftrightarrow$ at most one var. can be positive $(x_j \ge 0)$
- ▷ disjunctive constraints:

  want: $x \ge a$ or $y \ge b$ $(a, b \ge 0,\ x, y \ge 0)$

  $\Rightarrow$ introduce $\delta \in \{0,1\}$

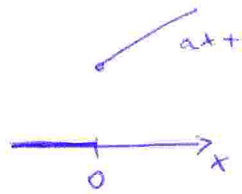
  and model $x \ge \delta a$

  $y \ge (1-\delta) b$
- ▷ conditional constraints:

  want: if $x > a$ then $y \ge b$ (else $y \ge 0$)

  $\Leftrightarrow$ $x \le a$ or $y \ge b$ $\Rightarrow$ disjunctive constr.
- ▷ fixed cost: (min)

  want: $c(x) = \begin{cases} ax + b & \text{if } x > 0 \\ 0 & \text{otherw.} \end{cases}$ $\xrightarrow{x \in \mathbb{R}^1}$



  $\Rightarrow$ introduce $\delta \in \{0,1\}$

  and $x \le u\delta$, u = upper bound on x

  then $c(x, \delta) = ax + b\delta$.

  [Why? $x > 0 \Rightarrow \delta = 1 \Rightarrow c(x) = ax + \delta$
  $x = 0 \Rightarrow \delta = 0$ in opt sol., bc. $a, b > 0 \Rightarrow c(x) = 0$]

[illegible]

→ ILP : • great flexibility in expressing discrete optimization problems ("its an art")

• for the price that it is much harder to solve than LP.

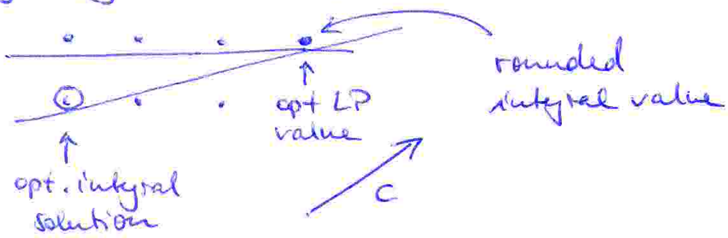
Note: simple rounding often no option!.

(i) doesn't make sense if integral var. is decision var.

e.g. shortest path: $x_{uv} = \begin{cases} 1 & uv \text{ in SP} \\ 0 & uv \text{ not in SP} \end{cases}$

(ii) generally may cause large errors

e.g.



Complexity of ILP

Thm. Solving ILP is NP-complete.

[To decide if there is a feasible integral solution is NP-complete.]

Pf. Show NP-hardness by reduction from SAT:

m clauses C_i and n vars. v_j:

$$\Phi = \underbrace{(x_{11} \vee x_{12})}_{C_1} \wedge \underbrace{(x_{21} \vee x_{22} \vee x_{23})}_{C_2} \wedge \dots \wedge \underbrace{(x_{m1} \vee x_{m2} \vee x_{m3} \dots)}_{C_m}$$

$x_{ij} \in \{v, \bar{v}\}$, $v \in \{v_1, \dots, v_n\}$ boolean vars.

Quest.: Does exist a satisfying truth assignment for Φ?

Construct ILP: $v_j \longrightarrow z_j \in \{0, 1\}$, $j = 1, \dots, n$

"false" "true"

For any clause C_i, $i = 1, \dots, m$, let $J^+(i)$ and $J^-(i)$ denote the set of indices of non-negated and negated vars. in C_i, resp.

$$\text{(ILP)}: \quad \sum_{j \in J^+(i)} z_j + \sum_{j \in J^-(i)} (1 - z_j) \geq 1 \quad , \quad i = 1, \dots, m$$

$$z_j \in \{0, 1\} \quad , \quad j = 1, \dots, n$$

Claim: $\exists$ satisfying truth assignment for Φ $\Leftrightarrow$ ILP has feas. sol.

"$\Rightarrow$" Given satisfying truth assignment v^* for Φ,
construct z^* : $z_j^* = \begin{cases} 1 & \text{if } v_j^* = \text{true} \\ 0 & \text{if } v_j^* = \text{false} \end{cases}$, $j = 1, \ldots, n$.

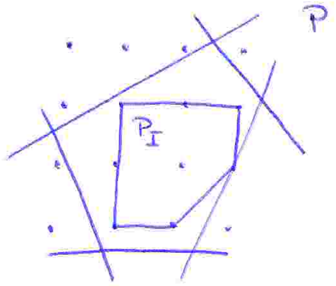
In each clause C_i at least one var. satisfied
$\Rightarrow$ correspond. ineq. in ILP are satisfied.

"$\Leftarrow$" z^* feas. solution to ILP $\Rightarrow$ construct $v_j^* = \begin{cases} \text{true}, & \text{if } z_j^* = 1 \\ \text{false}, & \text{otherw.} \end{cases}$
$\Rightarrow$ each clause has at least one var. satisfied $\Rightarrow$ Φ satisf. $\boxtimes$

ILP in NP? $\rightarrow$ 0/1 - ILP yes, because given solution x^* has polyn. size and can be verified in polyn. time $\Rightarrow$ ok.
$\rightarrow$ in general? From Ellipsoid method follows that the values x_j of an opt. sol. cannot be too large ($x_j \le 4^{nL}$ ... [LN]) $\Rightarrow$ ok.

$\square$

Integer hull of a polyhedron



- set of feasible integral solutions
  $\{x \mid Ax \le b, \ x \in \mathbb{Z}^n\}$
- note that $P := \{x \mid Ax \le b\}$ is a polyhedron

Let $P_I = \{x \mid Ax \le b\}_I$ be the convex hull of the integral vectors in P. (= "integer hull of P")

$\Rightarrow$ Can write ILP as $\max\{c^T x \mid x \in P_I\}$ where $P_I = \{x \mid Ax \le b\}$.

But not clear how to express in lin. ineq..

Proposition

(i) P bounded $\Rightarrow$ P_I polyhedron ($\{x \mid Ax \le b, x \in \mathbb{Z}^n\}$ finite)

(ii) P unbounded, A, b rational $\Rightarrow$ P_I polyhedron

(iii) P unbounded, A, b arbitrary real, then P_I in general no polyhedron.

Ex.: $P = \{(x,y) \in \mathbb{R}^2 \mid y \le \sqrt{2}\, x\}$

Important question:

Under which conditions is $P = P_I$?

$\leadsto$ in this case ILP can be solved by solving LP relaxation, i.e.,

$$\begin{aligned} &\max\ c^T x \\ &\text{s.t. } Ax \le b \\ &\cancel{x \in \mathbb{Z}^n} \quad \text{(relax)} \end{aligned}$$

Definition: Polyhedron P is integral $\Leftrightarrow$ $P = P_I$.

Thm. A rational polyhedron is integral

$\Leftrightarrow$ $\max\{c^T x \mid x \in P\} \in \mathbb{Z}$ for any $c \in \mathbb{Z}^n$ (for which finite)

Pf. "$\Rightarrow$" ✓

"$\Leftarrow$" Suppose $\forall c \in \mathbb{Z}^n$: $\max\{c^T x \mid x \in P\} \in \mathbb{Z}$.

Let y be unique optimum to $\max\{c^T x \mid x \in P\}$ for some $c \in \mathbb{Z}^n$.

We may assume that $c^T y > c^T x + x_1 - y_1$, $\forall x \in P$, $x \neq y$.

(Otherw. multiply c with some large integer.)

$\Rightarrow$ y is also optimal to $\max\{\bar{c}^T x \mid x \in P\}$ for $\bar{c} := (c_1 + 1, c_2, \ldots, c_n)^T$.

$$\Rightarrow \underbrace{\bar{c}^T y}_{\in \mathbb{Z}} = \underbrace{c^T y}_{\in \mathbb{Z}} + y_1 \quad \Rightarrow \quad y_1 \in \mathbb{Z}.$$

Repeat argument for all y_i $\Rightarrow$ $y \in \mathbb{Z}^n$

□

<u>Thm.</u> Let P be rational polyhedron. The following are equivalent:

(a) P is integral.

(b) Each non-empty face of P contains integral vectors.

(c) Each min. face of P contains integral vectors.

(d) $\max\{c^T x \mid x \in P\}$ is attained by an integral vector for each $c \in \mathbb{R}^n$ for which the max is finite.