Total Dual Integrality

Def. $Ax \le b$ is called totally dual integral (TDI) if for each $c \in \mathbb{Z}^n$ with $\max\{c^T x \mid Ax \le b\} < \infty$ holds:

$$\max\{c^T x \mid Ax \le b\} = \min\{y^T b \mid y^T A = c^T, y \ge 0, \underline{y \in \mathbb{Z}^m}\}.$$

$\rightsquigarrow$ gives primal integrality "for free".

Cor. Let $A \in \mathbb{Q}^{m \times n}$, $\underline{b \in \mathbb{Z}^m}$, and $Ax \le b$ TDI $\Rightarrow \{x \mid Ax \le b\}$ integral.

Pf. Def + Thm.
($b \in \mathbb{Z}^m \Rightarrow y^T b$ with $y \in \mathbb{Z}^m$ is integral again)

Application / Example

Given: directed graph $D = (V, A)$, $s, t \in V$
$\mathcal{P} = \{$ s-t paths in $D\}$, $c : A \to \mathbb{R}_+$

Task: Assign weights $y_a \ge 0$, $a \in A$, s.t. weight on any s-t path is at least 1.
Obj.: $\min \sum_{a \in A} c(a) y_a$

LP formulation:

$$\left.\begin{aligned} \min \sum_{a \in A} c(a) y_a & \\ \text{s.t.} \sum_{a \in P} y_a \ge 1 &, \quad P \in \mathcal{P} \\ y_a \ge 0 &, \quad a \in A \end{aligned}\right] \text{polyhedron } Q$$

Claim: Q is integral because it is TDI.

dual:

$$\begin{aligned} \max \sum_{P \in \mathcal{P}} 1 \cdot x_P & \\ \text{s.t.} \sum_{P : a \in P} x_P \le c(a) &, \quad \forall a \in A \\ x_P \ge 0 &, \quad \forall P \in \mathcal{P} \end{aligned}$$

$\to$ This is a formulation of the max s-t-flow problem. If the "capacities" $c(a)$, $a \in A$, are integral, then there exists an integral max flow.

$\Rightarrow$ Coroll. above implies that Q is integral.

$\square$

Remark: TDIness is <u>not</u> a property of polyhedra but a property of lin. ineq. systems. There can be two diff. descriptions of a particular polyhedron by ineq. systems s.t. one is TDI and the other one is not.

Ex.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

[TDI] [not TDI]

Generally TDI systems contain more constraints than necessary for just defining the polyhedron.

$\rightsquigarrow$ Adding valid ineq. does not destroy TDIness:

Prop. If $Ax \le b$ is TDI and $a^T x \le \beta$ is valid ineq. for $\{x \mid Ax \le b\}$, then $Ax \le b$, $a^T x \le \beta$ is also TDI.

Pf. For $c \in \mathbb{Z}^n$:

$$(*) = \min\{y^T b \mid y^T A = c^T, y \ge 0\} \overset{(\text{Dual})}{=} \max\{c^T x \mid Ax \le b\}$$

$$= \max\{c^T x \mid Ax \le b, \ a^T x \le \beta\}$$

$$\overset{(\text{dual})}{=} \min\{y^T b + \gamma\beta \mid y^T A + \gamma a^T = c^T, \ y, \gamma \ge 0\}$$

$$= (**)$$

$(*)$ has opt. integral solution y^* since $Ax \le b$ is TDI.

$\Rightarrow$ $(**)$ has opt. integral solution $(y^*, 0) = (y, \gamma)$.

$\square$

Cor.

- $Ax \le b, x \ge 0$ is TDI if $\min\{y^T b \mid y^T A \ge c, y \ge 0\}$ has integral opt. solution y for each $c \in \mathbb{Z}^n$ for which min is finite.
- $Ax = b, x \ge 0$ is TDI if $\min\{y^T b \mid y^T A \ge c\}$ has integral opt. solution y for each $c \in \mathbb{Z}^n$ for which min is finite.

⇝ Existence of TDI description with integral A?

<u>Thm.</u> For each rational polyhedron P, there exists a rational TDI-system $Ax \leq b$ with A integral and $P = \{x \mid Ax \leq b\}$. In particular, $b \in \mathbb{Z}^m \iff P$ integral.

⇝ Connection between TDI system and integral polyhedra.

⇒ "procedure" to prove integrality of polyhedra (justified by prev. Thm.)

- Find appropriate defining system $Ax \leq b$ with A, b integral.
- Prove $Ax \leq b$ TDI.
- From Thm. follows that $\{x \mid Ax \leq b\}$ integral.

must exist, but might be difficult to find!

$\boxed{\text{Question}}$ Are there matrices A s.t. $Ax \leq b$, $x \geq 0$ is TDI for each $b \in \mathbb{Z}^m$?

⇝ <u>totally unimodular matrices</u>