Totally unimodular matrices

Def. A matrix A is totally unimodular if each subdeterminant of A is $0, +1, -1$. Clearly, $A \in \{0,1,-1\}^{m\times n}$.
(A subdeterminant of $A \in \mathbb{Z}^{m\times n}$ is $\det B$ for some square submatrix B of A obtained by choosing an appropriate number of rows and columns of A.)

Thm [Hoffman, Kruskal '56]
$A \in \mathbb{Z}^{m\times n}$ totally unimodular $\Leftrightarrow$ $P = \{x \mid Ax \le b, x \ge 0\}$ integral for any $b \in \mathbb{Z}^m$.

Pf. "$\Rightarrow$" Let A be TU, $b \in \mathbb{Z}^m$, and x a vertex in P.
x is solution of ~~some subsystem~~ $A'x = b'$ for some subsystem $A'x \le b'$ of $\begin{pmatrix} A \\ -I \end{pmatrix} x \le \begin{pmatrix} b \\ 0 \end{pmatrix}$, with $A' \in \{0,1,-1\}^{n\times n}$ non-singular. ($\Rightarrow \det A' \neq 0$)
A TUM $\Rightarrow |\det A'| = 1$ $\Rightarrow$ x is integral.

Cramers rule: $x_i = \dfrac{\det(A'')}{\det A'}$ integral for integral $b \to A''$ int.

"$\Leftarrow$" Suppose all vertices of P are integral $\forall b \in \mathbb{Z}^m$.
Let $A' \in \mathbb{Z}^{k\times k}$ be non-sing. submatrix of A.
To show $|\det A'| = 1$. W.l.o.g.

$$A = \begin{pmatrix} A' & * \\ * & * \end{pmatrix}.$$

$$(A \; I_m) = \begin{pmatrix} \underbrace{\begin{matrix} A' \\ * \end{matrix}} & \begin{matrix} * \\ * \end{matrix} & \begin{matrix} I_k \\ 0 \end{matrix} & \underbrace{\begin{matrix} 0 \\ I_{m-k} \end{matrix}} \end{pmatrix}$$

$$B \in \mathbb{Z}^{m\times m}$$

$$\det A' = \det B.$$

To show $|\det A'| = |\det B| = 1$, prove that B^{-1} integral.

[Why? $B \cdot B^{-1} = 1 \Rightarrow \det B \cdot \det(B^{-1}) = 1$

B integral $\Rightarrow \det B$ integral $\Rightarrow$ if B^{-1} integral, then $|\det B| = |\det(B^{-1})| = 1$]

Let $i \in \{1, \dots, m\}$ and show that $B^{-1} \cdot e_i \in \mathbb{Z}^m$.

Choose $y \in \mathbb{Z}^m$ s.t. $z := y + B^{-1} e_i \geq 0$.

Then $b := Bz = By + e_i$ is integral.

Add zeros to $z \longrightarrow z'$ with $(A\ I_m)\, z' = Bz = b \in \mathbb{Z}^m$.

$$(A\, I_m)\, z' = \begin{pmatrix} A' & * & I_k & 0 \\ * & * & 0 & I_{m-k} \end{pmatrix} \underbrace{\begin{pmatrix} z_1 \\ 0 \\ 0 \\ z_2 \end{pmatrix}}_{z'} \begin{array}{l} \} k \ \} = z'' \\ \\ \\ \} m-k \end{array} = b.$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Let z'' consist of first n entries of z'.

$\Rightarrow z'' \in P$:

$$m+n \left\{ \begin{array}{l} k\{ \\ \\ \\ \end{array} \begin{pmatrix} \overset{k}{\widetilde{A'}} & * \\ * & * \\ \multicolumn{2}{c}{-I_n} \end{pmatrix} \cdot z'' \right. \leq \begin{pmatrix} b \\ 0 \end{pmatrix} \begin{array}{l} \}m \\ \}n \end{array}$$

Furthermore, z'' satisfies $\uparrow$ with equality for the first k rows and the last $n-k$ rows <u>and</u> they are lin. independent.

$\Rightarrow z''$ is a vertex of $P \Rightarrow z'' \in \mathbb{Z}^n \Rightarrow z' \in \mathbb{Z}^{m+n}$ (consists of z'' and slack)

$\Rightarrow z$ is integral $\Rightarrow B^{-1} e_i = z - y$ is integral.

$\square$

Thm. Let $A \in \mathbb{Z}^{m\times n}$. The following statements are equiv.:

(i) A is totally unimodular.

(ii) $\forall b \in \mathbb{Z}^m, \forall c \in \mathbb{Z}^n$:
$\max\{c^Tx \mid Ax \le b, x \ge 0\} = \min\{y^Tb \mid y^TA \ge c^T, y \ge 0\}$
have integral solutions x and y (if finite).

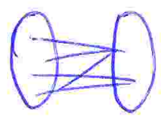
(iii) $Ax \le b, x \ge 0$ is TDI for all $b \in \mathbb{R}^m$.

(iv) $\forall R \subseteq \{1,\dots,m\}$ $\exists$ partition $R = R_1 \dot\cup R_2$ (disjoint):

$$\sum_{i\in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{-1,0,1\}, \quad \forall j = 1,\dots,m.$$

Coroll.: The node-edge incidence matrix of an undirected graph is totally unimodular iff graph is bipartite.

$\leadsto$ follows directly from (iv) $\Leftrightarrow$ (i).



$$\begin{array}{c|c|c} & e_1\ e_2 \cdots e_n & \\ \hline v_1, v_2, \vdots & 0 & R_1 \\ \hline v_m & 0 & R_2 \end{array}$$

resp. any subset of nodes

Thm. Let $A \in \{0,1,-1\}^{m\times n}$, where each column has at most one $+1$ and at most one -1. Then A is TU.

Pf. Let N be $k\times k$ submatrix of A. Induction on k:

$k=1$: $\det N \in \{0,1,-1\}$ ✓

$k \ge 2$: (i) N has at least one column with at most one non-zero entry

$\Rightarrow$ expand determinant along this column
$\Rightarrow \det N = \pm 1 \cdot \det N'$ for $(k-1)\times(k-1)$ matrix
$\Rightarrow \det N \in \{0,1,-1\}$ by ind. hypoth.

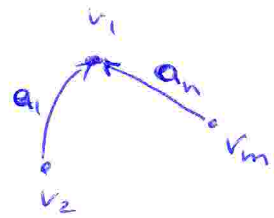
(ii) N has all columns with more than one non-zero entry (one $+1$, one -1).
Then sum of all rows gives $(0,\dots,0)$
$\Rightarrow$ lin. dependent $\Rightarrow \det N = 0$.

□

Coroll.: The node-edge incidence matrix of any digraph is TU.

$\leadsto$ follows directly from prev. Thm.

$$D = (V, A) \quad : \quad \begin{array}{c|c} & \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_m \\ \end{array} & \begin{pmatrix} +1 & & & +1 \\ -1 & & & 0 \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 & & & -1 \end{pmatrix} \end{array}$$



Theorem: A matrix $A \in \{0,1\}^{m \times n}$ has the consecutive ones property (along columns), if in every column the 1's appear consecutively (assuming some lin. ordering of rows of A).
Any matrix with the consecutive ones property is TU.

Pf. [Homework]