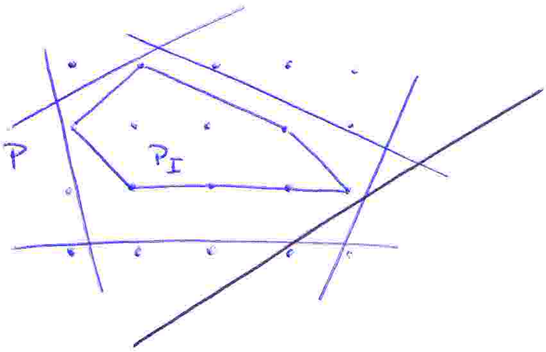
⇝ previous part: - considered integral polyhedra, but in general $P_I \subset P$.
- never used particular cost functions

Cutting planes and cutting plane algorithms

Idea: Cut off parts of P that do not contain integral points s.t. resulting set is polyhedron P', $P \supset P' \supset P_I$.



"cutting plane" = hyperplane

"separating hyper-/cutting plane" = cutting plane that cuts off some $x \in P \setminus P_I$.

Generic approach

Input: ILP $\max\{c^T x \mid x \in P_I\}$

Output: opt. solution to ILP

Method: Repeat
- solve LP $\max\{c^T x \mid x \in P\}$ ⇝ x^* opt. solution
- If x^* integral, then return x^*; else:
- Find a separating cutting plane that cuts off x^* from P and is valid for P_I. Let H be corresponding half space.
- Let $P := P \cap H$.

Questions:

(1) How to prove that an ineq. is valid for P_I?
⇝ "cutting plane proofs"

(2) Does the cutting plane method terminate?
⇝ positive answer for cutting planes of particular type.
(Gomory-Chvátal cuts and Chvátal rank)

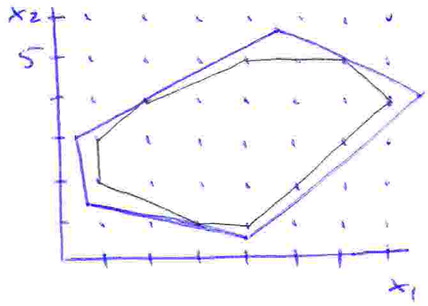
(3) Given $x^* \in P \setminus P_I$, how to find separating hyperplane?
⇝ problem dependent (general ex: Gomory's algorithm)

Cutting plane proofs

Want to prove that $c^T x \le \delta$ is valid for all integral solutions in $Ax \le b$. (W/out integrality ⇝ Farkas Lemma!)

Example

$$
\begin{aligned}
2x_1 + 3x_2 &\le 27 && (1)\\
2x_1 - 2x_2 &\le 7 && (2)\\
-6x_1 - 2x_2 &\le -9 && (3)\\
-2x_1 - 6x_2 &\le -11 && (4)\\
-6x_1 + 8x_2 &\le 21 && (5)
\end{aligned}
$$



$\boxed{x_2 \le 5}$

Obvious from picture, but how to derive from ineq. syst.?

a) $\frac{1}{2} \times (5) \Rightarrow -3x_1 + 4x_2 \le 21/2$

$\Rightarrow -3x_1 + 4x_2 \le \lfloor 21/2 \rfloor = 10$ (6)

New ineq. (6) is valid for P_I since (x_1, x_2) is integral.

b) $2 \times (6) + 3 \times (1)$:

$$
\begin{aligned}
(6')\quad -6x_1 + 8x_2 &\le 20\\
(1')\quad 6x_1 + 9x_2 &\le 81\\
\hline
17x_2 &\le 101
\end{aligned}
\quad \Rightarrow \quad x_2 \le \left\lfloor \frac{101}{17} \right\rfloor = 5
$$

In general: Given $Ax \le b$. Valid ineq. are of the form $y^T A x \le \lfloor y^T b \rfloor$ with $y \ge 0$ and $y^T A$ integral.

Such ineq. are called Gomory-Chvatal cutting planes.

Def. Let $Ax \le b$ be a system of lin. ineq., and $c^T x \le \delta$ an ineq.
A sequence of lin. ineq.

$$c_1 x \le \delta_1, \; c_2 x \le \delta_2, \; \dots, \; c_m x \le \delta_m$$

is called cutting plane proof of $c^T x \le \delta$ (from $Ax \le b$) if:

- $c_1, c_2, \dots, c_m$ integral
- $c_m = c$, $\delta_m = \delta$
- $\forall i = 1, \dots, m$: $c_i x \le \delta_i'$ is a non-negative lin. combination of ineq. $Ax \le b, c_1 x \le \delta_1, \dots, c_{i-1} x \le \delta_{i-1}$ for δ_i' with $\lfloor \delta_i' \rfloor = \delta_i$.

Thm. [Existence]

Let $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ be bounded and non-empty.

(i) If $P_I \neq \emptyset$ and $c^T x \le \delta$ is valid for P_I (c integral), then there is a cutting plane proof of $c^T x \le \delta$ from $Ax \le b$.

(ii) If $P_I = \emptyset$, then there is a cutting plane proof of $0x \le -1$ from $Ax \le b$

Thm. [finite length]

Let $P = \ldots$ as above.

(iii) If $P_I \neq \emptyset$ and $c^T x \le \delta$ is valid for P_I (c integral), then $c^T x \le \delta$ has a cutting plane proof of finite length.

$\leadsto$ Shortest possible proof for a valid ineq. and a particular syst. ?

The Chvátal rank

Assume we don't add single cuts sequentially, but in "waves". (= all currently feas. Gomory-Chvátal-Cuts at once)

Def. Let $P = \{x \mid Ax \le b\}$ be rational polyhedron. Adding all Gomory-Chvátal cuts $y^T Ax \le \lfloor y^T b \rfloor$ with $y \ge 0$, $y^T A$ integral to P yields the Chvátal Closure P'.

Thm. If P is a rational polyhedron, then P' is also a rational polyhedron

$\leadsto$ special case TDI systems:

Thm. Let $P = \{x \mid Ax \le b\}$ be a polyhedron with $Ax \le b$ TDI, A integral b rational. Then, $P' = \{x \mid Ax \le \lfloor b \rfloor\}$.

Pf. Let $P \neq \emptyset$, othw. trivial. Obviously $P' \subseteq \{x \mid Ax \le \lfloor b \rfloor\}$.

To show $\{x \mid Ax \le \lfloor b \rfloor\} \subseteq P'$. Let $y \ge 0$ with $y^T A$ integral.

To show: $\forall x$ with $Ax \le \lfloor b \rfloor$: $y^T Ax \le \lfloor y^T b \rfloor$.

$$y^T b \ge \max\{\underbrace{y^T A}_{\text{integral}} x \mid Ax \le b\} \underset{\text{duality}}{=} \min\{v^T b \mid v \ge 0, v^T A = y^T A\} = v$$

$Ax \le b$ TDI $\Rightarrow$ min attained for integral v^*. Then

$$\underline{y^T A x} = \underline{v^{*T} A} x \underset{Ax \le \lfloor b \rfloor}{\le} v^* \lfloor b \rfloor \le \lfloor v^* b \rfloor \le \lfloor y^T b \rfloor.$$

$\square$

Construction of Chvátal closures can be iterated naturally:

$$P = P^{(0)} \supseteq P^{(1)} \supseteq \dots \supseteq P^{(k)} = P_I .$$

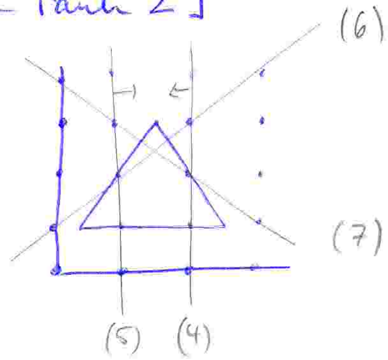
<u>Thm.</u> If P is a rational polyhedron (or a bounded polyhedron), then there exists a $k \in \mathbb{N}$ s.t. $P^{(k)} = P_I$.

<u>Def.</u> The smallest k s.t. $P^{(k)} = P_I$ is called <u>Chvátal rank</u>.

- kind of complexity measure for integer hull of polyhedra
- can be large!
- for several combinat. optimization problems and particular constraints, Chvátal rank (resp. upper bound) is known
- A totally unimodular $\Rightarrow$ Chv. rank $= 0$

<u>Example</u> [polyhedron with Chvátal rank 2]

$$\begin{array}{rrll} -2x_1 + x_2 & \le & 0 & (1) \\ 2x_1 + x_2 & \le & 6 & (2) \\ -x_2 & \le & -1 & (3) \end{array}$$



$\triangleright$ First Chvátal closure $P^{(1)}$:

$y^T = (0, \frac{1}{2}, \frac{1}{2}) \rightsquigarrow x_1 \le \frac{5}{2} \Rightarrow x_1 \le 2$ (4)

$y^T = (\frac{1}{2}, 0, \frac{1}{2}) \rightsquigarrow -x_1 \le -\frac{1}{2} \Rightarrow -x_1 \le -1$ (5)

$y^T = (\frac{5}{6}, \frac{1}{3}, \frac{1}{6}) \rightsquigarrow -x_1 + x_2 \le \frac{11}{6} \Rightarrow -x_1 + x_2 \le 1$ (6)

$y^T = (\frac{1}{3}, \frac{5}{6}, \frac{1}{6}) \rightsquigarrow x_1 + x_2 \le \frac{29}{6} \Rightarrow x_1 + x_2 \le 4$ (7)

<u>Note</u> $x_2 \le 2$ cannot be derived from $Ax \le b$ ((1)–(3))

Check general Gomory-Chvátal Cut $y^T A x \le \lfloor y^T b \rfloor$:

$$\underbrace{(-2y_1 + 2y_2)}_{=0} \underbrace{x_1}_{\ge 1} + \underbrace{(y_1 + y_2 - y_3)}_{=1} x_2 \le \lfloor \underbrace{6y_2 - y_3}_{=2} \rfloor$$

$\Rightarrow y_1 = y_2$ $\quad\Rightarrow y_3 = 2y_2 - 1$ $\quad\Rightarrow \lfloor 6y_2 - (2y_2 - 1) \rfloor = \lfloor 4y_2 + 1 \rfloor = 2$

$\Leftrightarrow 1 < 4y_2 < 2 \Rightarrow y_2 < \frac{1}{2} \Rightarrow y_3 < 0$ ↯

▷ 2nd Chvatal closure $P^{(2)}$

$x_2 \leq 2$ follows from (6) and (7) with $y^T = (\frac{1}{2}, \frac{1}{2})$.

Gomory's cutting plane algorithm

Repeat

1) Solve LP relaxation by Simplex.
   (Bring LP in standard form $Ax = b$, $x \geq 0$.)
   Let x^* be opt. basic feas. solution and B an associated basis.
   If x^* integral $\Rightarrow$ done.
2) From opt. tableau we obtain:

$$x_B + B^{-1}A_N x_N = B^{-1}b.$$

Let $\bar{a}_{ij} = (B^{-1}A_j)_i$ and $\bar{a}_{i0} = (B^{-1}b)_i$.

Consider a row i from tableau with $\bar{a}_{i0}$ fractional:

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{a}_{i0},$$

and add new constraint

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{a}_{i0} \rfloor. \qquad (*)$$

Why is $(*)$ a valid constraint?

From $x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{a}_{i0}$ with $x \geq 0$ follows

that $x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \bar{a}_{i0}$ is valid.

$\Rightarrow x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{a}_{i0} \rfloor$ valid for integral x.

Thm. The Gomory cutting plane algorithm terminates after a finite # of iterations when using the lexicographic dual simplex and always choosing i with min index.

Example

$$\min \quad x_1 - 2x_2$$
$$\text{s.t.} \quad -4x_1 + 6x_2 \le 9$$
$$x_1 + x_2 \le 4$$
$$x_1, x_2 \in \mathbb{Z}^+$$

(i) Simplex on standard form (add slack var. $x_3, x_4 \ge 0$) gives $x^1 = (\frac{15}{10}, \frac{25}{10})$.

One of the equations in tableau is $x_2 + \frac{1}{10}x_3 + \frac{1}{10}x_4 = \frac{25}{10}$.

$\Rightarrow$ Gomory cut : $\boxed{x_2 \le 2}$

Add constraint (with slack var.) : $x_2 + x_5 = 2$, $x_5 \ge 0$.

(ii) New simplex solution $x^2 = (\frac{3}{4}, 2)$.

Equation: $x_1 - \frac{1}{4}x_3 + \frac{6}{4}x_5 = \frac{3}{4}$

$\Rightarrow$ Gomory cut : $\boxed{x_1 - x_3 + x_5 \le 0}$

Substitute $x_5 = 2 - x_2$ and $x_3 = 9 + 4x_1 - 6x_2$ and add new constraint : $-3x_1 + 5x_2 + x_6 = 7$.

(iii) New simplex solution $x^3 = (1, 2) = x^*$ to ILP.

- Gomory's alg. itself not too practical. But in combination with other methods (Branch & bound) very powerfull.
- Cutting plane methods are part of many/all commercial LP solvers. (automatic generation to strengthen LP relat.)
- problem dependent $\rightarrow$ combinatorial structure $\rightarrow$ TSP: comb inequalities

Comb inequalities for TSP

Symmetric TSP on $G=(V,E)$: $x_e = \begin{cases} 1 & e \text{ is in tour} \\ 0 & \text{otherw.} \end{cases}$, $e \in E$

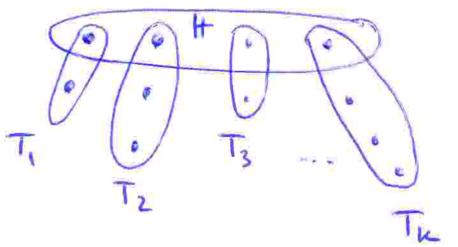
$$\min \sum_{e\in E} c_e x_e$$

$$\text{s.t.} \sum_{e\in\delta(v)} x_e = 2 \quad , \forall v \in V \qquad (1)$$

$$\sum_{e\in E(S)} x_e \leq |S|-1 \quad , \forall S \subseteq V,\ 2 \leq |S| \leq |V|-1 \qquad (2)$$

$$x_e \in \{0,1\} \quad , \forall e \in E \qquad (3)$$

- (2) is called subtour elimination constraint ; expon. many!
- LP relaxation: $\min \sum_{e\in E} c_e x_e$
  (1), (2)
  $x_e \geq 0 \quad , \forall e \in E$
- "Def." Comb with handle H and teeth $T_1, \dots, T_k$:



$H, T_i \subseteq V$, $i = 1,\dots,k$ with $k \geq 3$ and <u>odd</u>.

$|H \cap T_i| \geq 1$, $i=1,\dots,k$

$|T_i \setminus H| \geq 1$, $i = 1,\dots,k$

$T_i \cap T_j = \emptyset$, $i \neq j$

<u>Thm</u>: Let C be a comb with handle H and teeth $T_1, \dots, T_k$ for $k \geq 3$ and odd. Then any tour x satisfies:

$$\sum_{e\in E(H)} x_e + \sum_{i=1}^{k} \sum_{e\in E(T_i)} x_e \leq |H| + \sum_{i=1}^{k} (|T_i|-1) - \frac{k+1}{2} .$$

$\longrightarrow$

Pf.: Let x be feas. tour $\Rightarrow$ it satisfies (1) – (3).

Add up constraints:

(1) for all $v \in H$

(3) for all $e \in \delta(H) \setminus X$ where X are edges that belong to a tooth (take $-x_e \le 0$)

(2) for all T_i, $i = 1, \dots, k$

(2) for all $T_i \setminus H$, $i = 1, \dots, k$

(2) for all $T_i \cap H$, $i = 1, \dots, k$

$$\begin{aligned}
&\sum_{v \in H} \sum_{e \in \delta(v)} x_e &&= 2|H| \\
-&\sum_{v \in H} \sum_{\substack{e \in \delta(v) \\ \text{except} \\ e \in E(T_i)}} x_e &&\le 0
\end{aligned}
\quad \Bigg] \quad 2 \sum_{e \in E(H)} x_e \le 2|H|$$

$$\begin{aligned}
+&\sum_{i=1}^{k} \sum_{e \in E(T_i)} x_e &&\le \sum_{i=1}^{k} (|T_i| - 1) \\
+&\sum_{i=1}^{k} \sum_{e \in E(T_i \setminus H)} x_e &&\le \sum_{i=1}^{k} (|T_i \setminus H| - 1) \\
+&\sum_{i=1}^{k} \sum_{e \in E(T_i \cap H)} x_e &&\le \sum_{i=1}^{k} (|T_i \cap H| - 1)
\end{aligned}
\quad \Bigg] \quad \sum_{i=1}^{k} \sum_{e \in E(T_i)} x_e \le 2 \cancel{\sum_{i=1}^{k} (|T_i| - 1)} \le \sum_{i=1}^{k} (|T_i| - 1) - k$$

$$2 \sum_{e \in E(H)} x_e + 2 \sum_{i=1}^{k} \sum_{e \in E(T_i)} x_e \le 2|H| + 2 \sum_{i=1}^{k} (|T_i| - 1) - k \qquad | : 2$$

$$\Leftrightarrow \quad \sum_{e \in E(H)} x_e + \sum_{i=1}^{k} \sum_{e \in E(T_i)} x_e \le |H| + \sum_{i=1}^{k} (|T_i| - 1) - \frac{k}{2}$$

is a valid inequality for any comb $C(H, T_i)$.

Any integral solution x then satisfies the ineq. as claimed after rounding down $-\frac{k}{2}$ to $-\frac{k+1}{2}$ ($k \ge 3$ and odd).

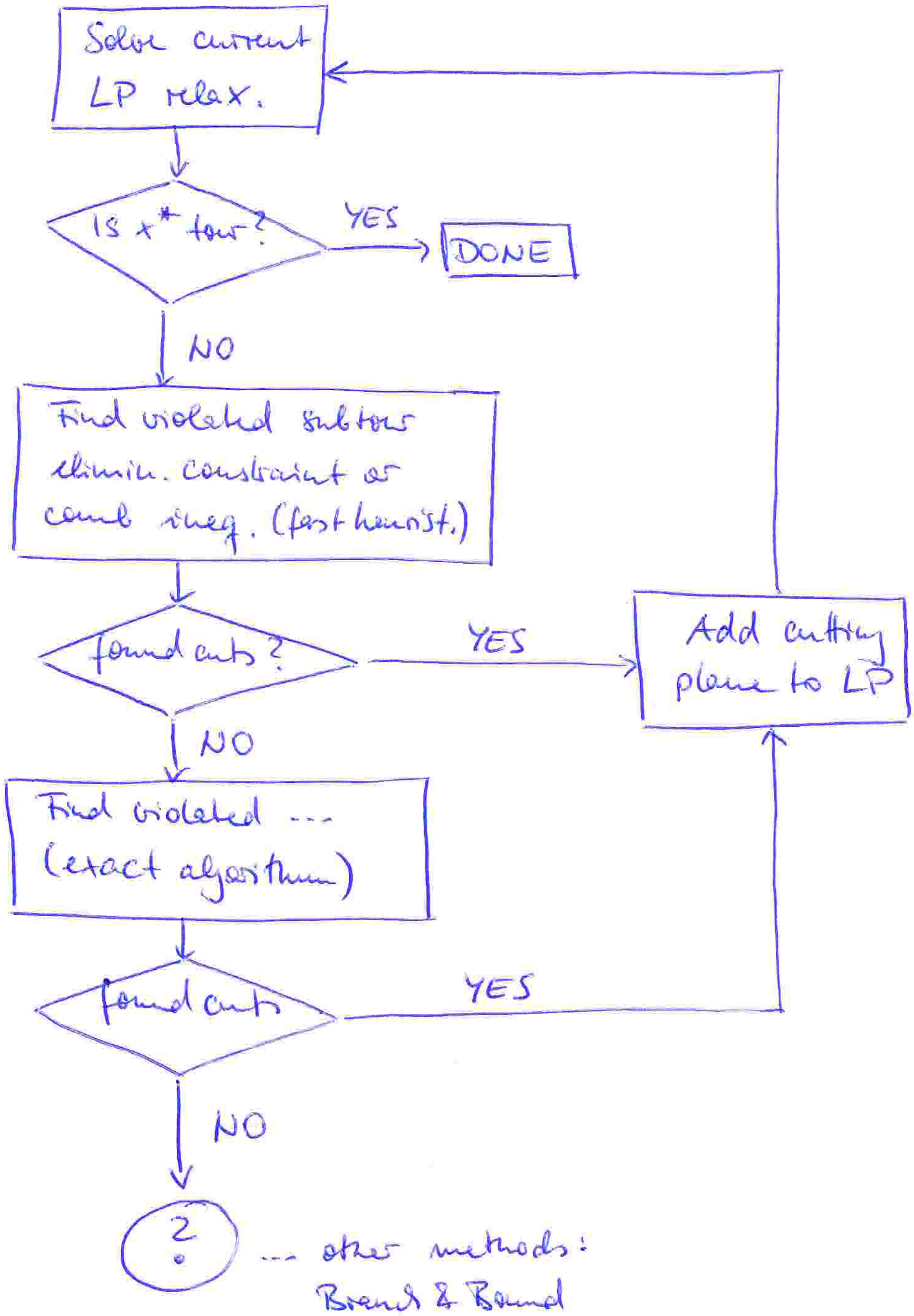
$=$ Gomory-Chvátal Cut.

$\square$

- no polyn. time separation alg. known that decides for a given non-negative vector x if it violates a comb ineq.

- exponential time separation alg. known and heuristics (may or may not find a violated ineq.)

Application / Solution method : begin with LP without subtour elimin. constraints (too many)



▷ Disadvantage of "pure" cutting plane algorithms: no feas. solution until alg terminates → combine with Branch & Bound Alg.