

Lecture 2 — April 14

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2.1 Basic notation

Let us review some basic definitions and notation of matrices and vectors.

Definition 2.1. A matrix $\mathbf{A} \in \mathcal{R}^{n \times m}$ is defined by numbers $a_{ij} \in \mathcal{R}$ where $1 \leq i \leq n$ and $1 \leq j \leq m$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Definition 2.2. The transpose of a matrix $\mathbf{A} \in \mathcal{R}^{n \times m}$ is the matrix $\mathbf{B} \in \mathcal{R}^{m \times n}$ such that $a_{ij} = b_{ji}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. We denote the transpose of \mathbf{A} by \mathbf{A}' .

Definition 2.3. The product of two matrices $\mathbf{A} \in \mathcal{R}^{n \times k}$ and $\mathbf{B} \in \mathcal{R}^{k \times m}$ is the matrix $\mathbf{C} \in \mathcal{R}^{n \times m}$ such that $c_{ij} = \sum_{p=1}^k a_{ip} b_{pj}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Always remember that to be able to multiply matrix \mathbf{A} and \mathbf{B} , the number of columns of \mathbf{A} must equal the number of rows of \mathbf{B} .

Definition 2.4. A vector $\mathbf{x} \in \mathcal{R}^n$ is just a $n \times 1$ matrix and we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{or} \quad x = (x_1, \dots, x_n).$$

By default vectors will be *column vectors* as above. An n dimensional *row vector* will be a matrix of dimension $1 \times n$, that is, the transpose of a column vector.

Definition 2.5. The dot product of two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ is $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$.

When viewing vectors $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ as matrices we sometimes write $\mathbf{x}'\mathbf{y}$ instead of $\mathbf{x} \cdot \mathbf{y}$.

2.2 Polyhedra

Definition 2.6. A polyhedron is the set $\{x \in \mathcal{R}^n : \mathbf{Ax} \geq \mathbf{b}\}$ defined by a matrix $\mathbf{A} \in \mathcal{R}^{m \times n}$ and a vector $\mathbf{b} \in \mathcal{R}^m$.

We'll say that the polyhedron $P \subseteq \mathcal{R}^n$ is *bounded* if $P \subseteq [-k, k]^n$ for some $k > 0$, and *unbounded* otherwise.

Definition 2.7. Let \mathbf{a} be a non-zero vector in \mathcal{R}^n and $\mathbf{b} \in \mathcal{R}$. Then $\{\mathbf{x} \in \mathcal{R}^n : \mathbf{a}'\mathbf{x} = \mathbf{b}\}$ is called a hyperplane and $\{\mathbf{x} \in \mathcal{R}^n : \mathbf{a}'\mathbf{x} \leq \mathbf{b}\}$ is called a halfspace.

Notice that in \mathcal{R}^2 a hyperplane is a line and a halfspace is a halfplane.

Halfspaces are the building blocks of polyhedra: A polyhedron is nothing but the intersection of a bunch of halfspaces.

Definition 2.8. A set $S \subseteq \mathcal{R}^n$ is convex if for all $\mathbf{x}, \mathbf{y} \in S$ and $\lambda \in [0, 1]$ the vector $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ belong to S as well.

Theorem 2.9. Any polyhedron is a convex set.

Proof: It is not hard to show that convex sets are closed under intersections and that halfspaces are convex. It follows then that polyhedra are always convex. \square

2.3 Extreme points, vertices, and basic feasible solutions

Definition 2.10. Let P be a polyhedron. A vector $\mathbf{x} \in P$ is an extreme point of P if $\nexists \mathbf{y}, \mathbf{z} \in P$ such that $\mathbf{y}, \mathbf{z} \neq \mathbf{x}$ and \mathbf{x} is a convex combination of \mathbf{y} and \mathbf{z} .

Definition 2.11. Let P be a polyhedron. A vector $\mathbf{x} \in P$ is a vertex of P if $\exists \mathbf{c}$ such that $\mathbf{c} \cdot \mathbf{x} < \mathbf{c} \cdot \mathbf{y}$ for all $\mathbf{y} \in P - \mathbf{x}$.

Given a polyhedron $P = \{\mathbf{x} \in \mathcal{R}^n : \mathbf{a}_i \cdot \mathbf{x} \geq b_i \text{ for } i = 1, \dots, m\}$ and a vector $\mathbf{x} \in \mathcal{R}^n$, we will say that the i th constraint is active if $\mathbf{a}_i \cdot \mathbf{x} = b_i$.

Definition 2.12. Let P be a polyhedron. Then $\mathbf{x} \in P$ is a basic feasible solution if the set of active constraints at \mathbf{x} has full rank.

Theorem 2.13. Let P be a non-empty polyhedron and $\mathbf{x} \in P$. The following statements about \mathbf{x} are equivalent:

1. \mathbf{x} is a vertex.
2. \mathbf{x} is an extreme point.
3. \mathbf{x} is a basic feasible solution.

2.4 Polyhedra in standard form

It will be convenient for our algorithm for deal with polyhedra in standard form. Such polyhedra are defined by a matrix $\mathbf{A} \in \mathcal{R}^{m \times n}$ having m linearly independent rows and a vector $\mathbf{b} \in \mathcal{R}^m$.

$P = \{\mathbf{x} \in \mathcal{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}\}$ is a polyhedron in standard form.

Definition 2.14. Given a polyhedron $P \subseteq \mathcal{R}^n$ in standard form, the set $B \subseteq [1, n]$ is a basis if $|B| = m$ and \mathbf{A}_B (the columns of \mathbf{A} indexed by B) have full rank. The vector \mathbf{x} defined as $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b}$ and $\mathbf{x}_{[1, n] \setminus B} = \mathbf{0}$ is the basic solution induced by the basis B .

Notice that if the basic solution \mathbf{x} also happens to be feasible it agrees with our previous definition of basic feasible solution.