This part of the course will follow the book Approxmation Algorithms by Vijay V. Vazirani [Vaz00]. I will cover.

1 The Set Cover Problem

The Greedy Algorithm, pages 15 to 17: Proof of Lemma 2.3: Vijay writes:

> In any iteration, the leftover sets of the optimal solution can cover the remaining elements at a cost of at most OPT. Therefore, among these sets, there must be one of having cost-effectiveness of at most $\mathrm{OPT}/|\overline{C}|$, where $\overline{C}$ is the set of uncovered elements.

I had a hard time understanding this sentence. It is supposed to prove

$$\min_{S\in\mathscr{S}:} \frac{c(S)}{|S\cap\overline{C}|} \le \frac{\mathrm{OPT}}{|\overline{C}|}.$$

An extended argument is as follows.

$$\begin{aligned}
\frac{\mathrm{OPT}}{|\overline{C}|} = \frac{\sum_{S\in\mathrm{OPT}} c(S)}{|\overline{C}|} &= \frac{\sum_{S\in\mathrm{OPT}} \sum_{e\in S\cap\overline{C}} \frac{c(S)}{|S\cap\overline{C}|}}{|\overline{C}|} \\
&= \frac{\sum_{e\in\overline{C}} \sum_{S\in\mathrm{OPT}:e\in S} \frac{c(S)}{|S\cap\overline{C}|}}{|\overline{C}|} \\
&\ge \min_{e\in\overline{C}} \sum_{S\in\mathrm{OPT}:e\in S} \frac{c(S)}{|S\cap\overline{C}|} \\
&\ge \min_{e\in\overline{C}} \min_{S\in\mathrm{OPT}:e\in S} \frac{c(S)}{|S\cap\overline{C}|} \\
&\ge \min_{S\in\mathscr{S}:} \frac{c(S)}{|S\cap\overline{C}|}
\end{aligned}$$

Here is a simpler argument. Consider the leftover sets in OPT, i.e., their intersection with $\overline{C}$. Modify them as follows: as long as an element is covered by more than one element in OPT, remove the element from all but one. After this modification, the sets are disjoint and still cover $\overline{C}$. The modificiation worsens the cost-effectiveness of the sets. Since the sets are disjoint, the average cost of covering an element is $\mathrm{OPT}/|\overline{C}|$. Consider an element e which is covered at a cost no larger than average and the set S covering it. This set has a cost-effectiveness of no more than $\mathrm{OPT}/|\overline{C}|$.

Set Cover Via Dual Fitting, pages 108 to 110:

Rounding Applied to Set Cover, pages 118 to 120:

Set Cover via the Primal-Dual Scheme: pages 124 to 128:

The Steiner Problem and Variants: pages 27 to 30, 197 – 230:

2 EATCS-Award 2010

Have a look at `http://www.eatcs.org/index.php/eatcs-award/623`

References

[Vaz00] V.V. Vazirani. *Approximation Algorithms*. Springer, 2000.