Optimization **Summer 2010**

Lecture 3 — April 19

Lecturer: Julián Mestre

3.1 Representation of bounded polyhedra

Definition 3.1. *The convex hull of a set of vectors* $\mathbf{x^1}, \mathbf{x^2}, \ldots, \mathbf{x^k} \in \mathcal{R}^n$ *is the set of all convex combinations of these vectors:*

$$\mathrm{CH}(\mathbf{x^1}, \ldots, \mathbf{x^k}) = \left\{ \sum_{i=1}^{k} \lambda_i \, \mathbf{x^i} \;:\; \sum_{i=1}^{k} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \right\}.$$

The convex hull of a set of vectors is indeed convex; in fact, it is not difficult to show that it is the smallest convex set containing those vectors.

Theorem 3.2. *Let* $P \subseteq \mathcal{R}^n$ *be a bounded non-empty polyhedron. Then*

$$P = \mathrm{CH}(\{\mathbf{x} \in \mathcal{R}^\mathbf{n} \;:\; \mathbf{x} \text{ is an extreme points of } P\}).$$

Corollary 3.3. *Let* P *be a bounded non-empty polyhedron. Then the linear program*

$$\begin{aligned} \text{minimize} \quad & \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} \quad & \mathbf{x} \in P \end{aligned}$$

has an optimal solution in its set of extreme points for all $\mathbf{c}$.

3.2 Fourier-Motzkin elimination

Definition 3.4. *The projection of a vector* $\mathbf{x} = (x_1, \ldots, x_n)$ *onto its first* k *coordinates is defined as* $\pi_k(\mathbf{x}) = (x_1, \ldots, x_k)$*. The projection of* $S \subseteq \mathcal{R}^n$ *onto its first* k *coordinates is* $\pi_k(S) = \{\pi_k(\mathbf{x}) \;:\; \mathbf{x} \in S\}$.

Fourier-Motzkin's algorithm is a procedure to transform a representation of a polyhedron P into a representation for $\pi_{n-1}(P)$. We can obtain $\pi_k(P)$ by applying the basic procedure $n-k$ times.

Let $P = \{\mathbf{x} \in \mathcal{R}^n \;:\; \mathbf{a_i} \cdot \mathbf{x} \geq b_i \text{ for } i = 1, \ldots, m\}$ be a polyhedron. First we note that we can partition the constraints into three types: i) $a_{in} > 0$, ii) $a_{in} < 0$, and iii) $a_{in} = 0$. Let us use T_1, T_2, and T_3 to refer to the indices of the three kinds of constraints. The polyhedron P can then be written as

$$P = \left\{ x \in \mathcal{R}^n \;:\; \begin{array}{rll} x_n & \geq d_i + \mathbf{f_i} \cdot \pi_{n-1}(\mathbf{x}) & i \in T_1 \\ x_n & \leq d_i + \mathbf{f_i} \cdot \pi_{n-1}(\mathbf{x}) & i \in T_2 \\ 0 & \leq d_i + \mathbf{f_i} \cdot \pi_{n-1}(\mathbf{x}) & i \in T_3 \end{array} \right\}, \tag{3.1}$$

for suitable vectors $\mathbf{f_i} \in \mathcal{R}^{n-1}$ and scalars d_i where $i = 1, \ldots, m$.

Theorem 3.5. *For the polyhedron* P *defined in* (3.1) *we have*

$$\pi_{n-1}(P) = \left\{ \mathbf{y} \in \mathcal{R}^{n-1} \;:\; \begin{array}{rll} d_i + \mathbf{f_i} \cdot y & \leq d_j + \mathbf{f_j} \cdot \mathbf{y} & i \in T_1, j \in T_2 \\ 0 & \leq d_i + \mathbf{f_i} \cdot \mathbf{y} & i \in T_3 \end{array} \right\}$$

3.2.1 Some interesting consequences of the FM elimination

Theorem 3.6. *Let $P \subseteq \mathcal{R}^n$ be a polyhedron and $A \in \mathcal{R}^{m \times n}$ be a matrix. Then the set $\{\mathbf{Ax} : \mathbf{x} \in P\}$ is also a polyhedron.*

Two useful corollaries of this theorem are the fact that if P is a polyhedron then so is $\pi_k(P)$, and that the convex hull of a finite set of vectors is a polyhedron.

Theorem 3.7 (Farkas' lemma). *Let $\mathbf{A} \in \mathcal{R}^{m \times n}$ be a matrix and $\mathbf{b} \in \mathcal{R}^m$ be a vector. Exactly one of the following alternatives holds*

- *There is an $\mathbf{x} \in \mathcal{R}^n$ such that $\mathbf{Ax} \geq \mathbf{b}$, or*
- *There is a $\mathbf{y} \in \mathcal{R}^m$ such that $\mathbf{y} \geq 0$ and $\mathbf{y}'\mathbf{A} = \mathbf{0}$ and $\mathbf{y} \cdot \mathbf{b} > 0$.*