Optimization **Summer 2010**

Lecture 7 — May 3

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7.1 Linear programming duality

Let $\mathbf{A} \in \mathcal{R}^{m\times n}$ be a matrix with columns $\mathbf{A_1}, \ldots, \mathbf{A_n}$ and rows $\mathbf{a_1}, \ldots, \mathbf{a_m}$. With each linear program

$$
\begin{array}{lll}
\text{minimize} & \sum_j c_j x_j & \\
\text{subject to} & \mathbf{a_i'}\mathbf{x} \ge b_i & i \in M_1 \\
& \mathbf{a_i'}\mathbf{x} \le b_i & i \in M_2 \\
& \mathbf{a_i'}\mathbf{x} = b_i & i \in M_3 \\
& x_j \ge 0 & j \in N_1 \\
& x_j \le 0 & j \in N_2 \\
& x_j \text{ free} & j \in N_3
\end{array}
$$

we associate the dual program

$$
\begin{array}{lll}
\text{maximize} & \sum_i b_i y_i & \\
\text{subject to} & \mathbf{y'}\mathbf{A_j} \le c_j & j \in N_1 \\
& \mathbf{y'}\mathbf{A_j} \ge c_j & j \in N_2 \\
& \mathbf{y'}\mathbf{A_j} = c_j & j \in N_3 \\
& y_i \ge 0 & i \in M_1 \\
& y_i \le 0 & i \in M_2 \\
& y_i \text{ free} & i \in M_3
\end{array}
$$

Theorem 7.1 (weak duality). *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be a pair of feasible primal/dual solutions solutions of some linear program. Then* $\mathbf{c} \cdot \mathbf{x} \ge \mathbf{b} \cdot \mathbf{y}$ *holds.*

Theorem 7.2 (strong duality). *If a linear program has an optimal solution, so does its dual, and the value of both programs is the same.*

Theorem 7.3 (complementary slackness). *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be a pair of feasible primal/dual solutions. Both solutions are optimal if and only if*

$$
\begin{array}{rl}
p_i(\mathbf{a_i'}\mathbf{x} - b_i) = 0 & \forall\, i, \\
(c_j - \mathbf{y'}\mathbf{A_j})x_j = 0 & \forall\, j.
\end{array}
$$

7.2 Relation of primal and dual programs

Any given linear program can either be infeasible, feasible but have unbounded objective value, or feasible and have bounded objective value. The following table summarizes the possible combinations of primal-dual program pairs.

DUAL → PRIMAL ↓	FEASIBLE BOUNDED	FEASIBLE UNBOUNDED	INFEASIBLE
FEASIBLE BOUNDED	✓	✘	✘
FEASIBLE UNBOUNDED	✘	✘	✓
INFEASIBLE	✘	✓	✓