Computer Algebra
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To be handed in on May, 10th.
Discussion on May, 11th.

Exercise 4

Comparison of algebraic numbers

Given two (not necessarily square-free) polynomials $f, g \in \mathbb{Q}[x]$, show how to isolate their real roots and how to determine the corresponding multiplicities.
Let $\alpha, \beta \in \mathbb{R}$ be roots of f and g given by corresponding isolating intervals with rational endpoints. Formulate an algorithm to compute the *comparison predicate for* α *and* β, that is, to decide whether $\alpha < \beta$, $\alpha = \beta$ or $\alpha > \beta$.

Implementation of algebraic number comparison

Implement your algorithm from the previous exercise and compare pairwise

1. the smallest real root of $f = 12x^7 + 415x^6 - 3438x^5 + 667x^4 - 157x^3 + 2332x^2 - 437x + 691$ in $I_f = (0, 8)$
2. the largest real root of $g = 916x^5 + 1156x^4 + 1210x^3 - 569x^2 - 2946x + 144$ in $I_g = (-2, 2)$
3. the (unique) multiple root of $h = 998001x^4 - 2015982x^3 + 20080x^2 + 2015982x - 1018081$ in $I_h = (0, 2)$

The resultant in terms of polynomial roots

Let $f, g \in \mathbb{R}[x]$ not necessarily square-free, $\deg f = m$, $\deg g = n$. Prove that

$$\operatorname{Res}(f, g) = (\operatorname{LCF} f)^n \, (\operatorname{LCF} g)^m \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (\alpha_i - \beta_j),$$

where the α_i and β_j are the (complex) roots of f and g, respectively.
Hint: Exploit the fact that the resultant is uniquely determined by the three properties of Lemma 3.1.3.

Intersection of conics

A *conic section* $\mathcal{C}_f$ (or just *conic*) is a plane algebraic curve, defined as the vanishing locus of a bivariate polynomial $f \in \mathbb{Z}[x, y]$ of total degree 2:

$$f = \sum_{0 \le i+j \le 2} a_{i,j} x^i y^j \in \mathbb{Z}[x, y]$$

$$\mathcal{C}_f = \mathcal{V}_{\mathbb{R}}(f) = \{(x, y) \in \mathbb{R} \times \mathbb{R} : f(x, y) = 0\}$$

For two defining polynomials $f, g \in \mathbb{Z}[x, y]$ of conics, describe how to determine the intersection of the curves $\mathcal{C}_f$ and $\mathcal{C}_g$, that is, the set of roots common to both f and g.

$$\mathcal{C}_f \cap \mathcal{C}_g = \mathcal{V}_{\mathbb{R}}(f, g) = \{(x, y) \in \mathbb{R} \times \mathbb{R} : f(x, y) = g(x, y) = 0\}$$

Apply your algorithm on the following instances:

1. $f_1(x, y) = 4y^2 + 8y + x^2 - 4x - 1$
   $g_1(x, y) = y^2 + 2y + 4x^2 - 24x + 21$

2. $f_2(x, y) = 2y^2 + 7xy - x^2 - 2y + 3x - 5$
   $g_2(x, y) = 3y^2 - xy + x^2 + 2x - 7$

Hint: If $(x_0, y_0) \in \mathcal{C}_f \cap \mathcal{C}_g$ is a solution to $f = g = 0$, then $f(x_0, y)$ and $g(x_0, y) \in \mathbb{R}[y]$ share a common root.

Have fun with the solution!