Optimization I

Polyhedra, basic feasible solutions, optimality conditions

This assignment is due on May 10 in lecture. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, however, must be *written individually* without consulting someone elses solutions or any other source like the web.

1. (Exercise 2.22 in B&T) Let P and Q be polyhedra in $\mathbb{R}^n$. Let $P + Q = \{x + y | x \in P, y \in Q\}$.

   (a) Show that $P + Q$ is a polyhedron.

   (b) Show that every extreme point of $P + Q$ is the sum of an extreme point of P and an extreme point of Q.

2. ((Part of) exercise 2.10 in B&T) Consider the standard form polyhedron $P = \{Ax = b, x \geq 0\}$, and suppose we want to minimize $c^T x$ over $x \in P$ for some $c \in \mathbb{R}^n$. Suppose that A is an $m \times n$ matrix and its rows are linearly independent. For the following statements, state whether it is true or false. If true, provide a proof, if false, provide a counterexample.

   (a) If $n = m + 1$, then P has at most two basic feasible solutions.

   (b) At every optimal solution, no more than m variables may be positive.

3. (Exercise 2.3 in B&T) (**Basic feasible solutions in standard form polyhedra with upper bounds**) Complete the following statement and prove it is correct (*a brief sketch is sufficient*):

   > Consider a polyhedron defined by the constraints $Ax = b$ and $0 \leq x \leq u$. Assume that the $m \times n$ matrix A has linearly independent rows and that $u_i > 0$ for all i. A vector $x \in \mathbf{R}^n$ is a *basic feasible solution* if and only if we have $Ax = b$, and there exist indices $B(1), \ldots, B(m)$ such that
   >
   > (a) ...
   > (b) ...

4. (Part of) exercise 3.6 in B&T) (**Conditions for a unique optimum**) Let x be a basic feasible solution associated with some basic matrix **B**. Prove that if the reduced cost of every nonbasic variable is positive, then x is the unique optimal solution.

5. (Extra credit) Which question on homeworks 1 and 2 (excluding this one) did you like best? Which one did you like least?