Homework 8

Khaled Elbassioni
Anke van Zuylen

This assignment is **due on July 18** in class. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, however, must be written individually without consulting someone else's solutions or any other source like the web.

Problem 1: (Job shop scheduling) A factory consists of m machines $M_1, \ldots, M_m$, and needs to process n jobs every day. Job j needs to be processed once by each machine in the order $(M_{j(1)}, \ldots, M_{j(m)})$. Machine M_i takes time p_{ij} to process job j. A machine can only process one job at a time, and once a job is started on any machine, it must be processed to completion. The objective is to minimize the sum of the completion times of all the jobs. Provide an integer programming formulation for this problem.

Problem 2: In the undirected traveling salesman problem, we are given an undirected graph $G = (\mathcal{N}, \mathcal{E})$ with non-negative costs $(c_e : \; e \in E)$ on the edges, and seek a tour of minimum cost visiting all the nodes. Consider the following two IP formulations of the problem:

$$\begin{array}{ll} \text{minimize} & \sum_{e \in \mathcal{E}} c_e x_e \\ \text{subject to} & \sum_{e \in \delta(\{i\})} x_e = 2, \text{ for } i \in \mathcal{N}, \\ & \sum_{e \in \delta(S)} x_e \geq 2, \text{ for } S \subset \mathcal{N}, \; S \neq \emptyset, \mathcal{N}, \\ & x_e \in \{0,1\}, \text{ for } e \in \mathcal{E}, \end{array}$$

$$\begin{array}{ll} \text{minimize} & \sum_{e \in \mathcal{E}} c_e x_e \\ \text{subject to} & \sum_{e \in \delta(\{i\})} x_e = 2, \text{ for } i \in \mathcal{N}, \\ & \sum_{e \in E(S)} x_e \leq |S| - 1, \text{ for } S \subset \mathcal{N}, \; S \neq \emptyset, \mathcal{N}, \\ & x_e \in \{0,1\}, \text{ for } e \in \mathcal{E}, \end{array}$$

where $\delta(S) = \{\{i,j\} \in \mathcal{E} : \; i \in S \text{ and } j \notin S\}$ and $E(S) = \{\{i,j\} \in \mathcal{E} : \; i,j \in S\}$.

Let P_{tspcut} and P_{tspsub}, respectively, denote the polyhedra corresponding to the LP-relaxations of theses formulations. Prove that

$$P_{tspcut} = P_{tspsub}.$$

Problem 3: Consider the integer programming problem:

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 \\ \text{subject to} & -3x_1 + 4x_2 \leq 4 \\ & 3x_1 + 2x_2 \leq 11 \\ & 2x_1 - x_2 \leq 5 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \text{ integer.} \end{array}$$

Use a figure to answer the following questions.

(a) What is the optimal cost of the linear programming relaxation? What is the optimal cost of the integer programming problem?

(b) What is the convex hull of the set of all solutions to the integer programming problem?

(c) Illustrate how the Gomory cutting plane algorithm would work. Give the first cut.

(d) Solve the problem by branch and bound. Solve the linear programming relaxations graphically.

(f) Suppose you dualize the constraint $-3x_1 + 4x_2 \leq 4$. What is the optimal value Z_D of the Lagrangean dual?

(e) Suppose you dualize the constraint $2x_1 - x_2 \leq 5$. What is the optimal value Z_D of the Lagrangean dual?

Problem 4: (Branch and bound can take exponential time) Consider the integer programming problem

$$\begin{array}{ll} \text{minimize} & x_{n+1} \\ \text{subject to} & 2x_1 + 2x_2 + \cdots + 2x_n + x_{n+1} = n \\ & x_i \in \{0, 1\}. \end{array}$$

Show that any branch and bound algorithm that uses linear programming relaxations to compute lower bounds, and branches by setting a fractional variable to either zero or one, will require the enumeration of an exponential number of subproblems when n is odd.