The Probabilistic Method and Randomized Algorithms

Assignment 3 — Due on June 6, 2011

1. Consider a random binary sequence $(s_1, \ldots, s_n)$ of length n, where $\mathbf{P}(s_i = 1) = p$ and $\mathbf{P}(s_i = 0) = 1 - p$, independently for each $1 \le i \le n$. Let $k \ge 1$ be a fixed integer and let X_n denote the number of substrings $(s_j, s_{j+1}, \ldots, s_{j+k-1})$ of length k such that every bit has value 1.

   (a) Show that $p = p(n) = n^{-1/k}$ is a threshold function for $X_n > 0$. (The threshold function is defined in homework 2.)

   (b) Let $c \ge 0$ be a constant and assume $p = (c/n)^{1/k}$. Show that X_n is asymptotically distributed as a Poisson random variable with parameter c.

   (c) Estimate the asymptotic value of $\mathbf{P}(X_n = 0)$ for $p = o(n^{-1/(k+1)})$.

2. Let X_n denote the number of vertices in $\mathcal{G}(n,p)$ with degree 0. Let c be a fixed real number and assume

$$p = \frac{\ln n + c}{n},$$

   where the base of the logarithm is the natural number e. For any fixed integer $i \ge 0$, compute $\lim_{n\to\infty} \mathbf{P}(X_n = i)$.

3. Let $X_1, X_2, \ldots, X_n$ be n binary random variables. Let $X = \sum_{i=1}^n X_i$. Assume that for any subset $S \subseteq \{1, \ldots, n\}$,

$$\mathbf{P}\left(\wedge_{i\in S}(X_i = 1)\right) \le \prod_{i\in S} \mathbf{P}\left(X_i = 1\right).$$

   Derive a Chernoff bound for $\mathbf{P}\left(X \ge (1+\delta)\,\mathbf{E}\left(X\right)\right)$.
   Hint: Express $\mathbf{E}\left(e^{tX}\right)$ in terms of the Taylor series of $e^{(\cdot)}$ and then expand all terms to obtain an upper bound of $\mathbf{E}\left(e^{tY}\right)$ where $Y = \sum_{i=1}^n Y_i$ with $Y_1, \ldots, Y_n$ being n mutually independent random variables with $\mathbf{P}\left(Y_i = 1\right) = \mathbf{P}\left(X_i = 1\right)$.

4. Let $X_1, X_2, \ldots, X_n$ be n independent exponential random variables with non-negative parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, $\lambda_{\min} := \min_{1\le i\le n} \lambda_i$. Let $X := \sum_{i=1}^n X_i$ and $\mu = \mathbf{E}\left(X\right) = \sum_{i=1}^n \frac{1}{\lambda_i}$. Derive a Chernoff Bound for $\mathbf{P}\left(X \ge (1+\delta)\,\mathbf{E}\left(X\right)\right)$ (this bound may depend on $\lambda_{\min}$). *Hint:* If X_i is an exponential random variable with parameter λ_i, then the moment generating function is $\lambda_i/(\lambda_i - t)$ for any $t < \lambda_i$.