Topological Methods in Discrete Geometry

Summary of Lecture 2 MPI, Summer 2011

We prove topological Radon's theorem in any dimension d: given any map $f : \partial\Delta^{d+1} \to \mathbb{R}^d$, there exist two points $x_1, x_2 \in \partial\Delta^{d+1}$ with disjoint supports (i.e., the simplices that contain x_1 and x_2 are disjoint) and where $f(x_1) = f(x_2)$.

For technical reasons, we prove an equivalent statement: given any map $f : \Delta^{d+1} \to B^d$, there exist two points $x_1, x_2 \in \Delta^{d+1}$ with disjoint supports and where $f(x_1) = f(x_2)$.

Proof technique: As before, lets say, for contradiction, that there is a 'bad' map, i.e., a continuous map f with the property that for every two points x_1, x_2 with disjoint supports, $f(x_1) \neq f(x_2)$. Then if such a map f exists, we will extend it to a map on two derived spaces (with $\mathbb{Z}_2$-structure) such that it is a $\mathbb{Z}_2$-map. And then show, via calculation of their $\mathbb{Z}_2$-index, that such a $\mathbb{Z}_2$-map is impossible.

To realize this plan, we have to accomplish five things:

1. construct a derived space Y (with $\mathbb{Z}_2$-structure) for B^d,
2. construct a derived space X (with $\mathbb{Z}_2$-structure) for Δ^{d+1},
3. construct a $\mathbb{Z}_2$-map, say f_{join}, from X to Y
4. upper-bounding the $\mathbb{Z}_2$-index of Y
5. lower-bounding the $\mathbb{Z}_2$-index of X

The proof would work if $Ind_{\mathbb{Z}_2}(X) > Ind_{\mathbb{Z}_2}(Y)$. We now show each of these steps in order of $1, 4, 2, 5, 3$.

1. Space Y

We will take two copies of B^d, and embed them in $\mathbb{R}^{2d+2}$, and add line segments from every point x in the first copy to every point y in the second copy.

Formally, let $\psi_1 : B^d \to \mathbb{R}^{2d+2}$ be the function that maps each $x \in B^d$ to a point $f(x) \in \mathbb{R}^{2d+2}$. So $\psi_1(B^d)$ defines the first embedding of B^d, and similarly define $\psi_2(B^d)$ to be the second copy. We will have to be careful in the exact geometric embedding to ensure that

- no two line segments intersect. This would imply that each point can be written uniquely as $t \cdot \psi_1(x) + (1-t) \cdot \psi_2(y)$, where x, y are points of B^d, and $t \in [0,1]$ controls the position of this point lying on the segment $\overline{\psi_1(x)\psi_2(y)}$. Then each point can be written as the ordered pair (x, y, t), where x and y lie in B^d, and $t \in [0,1]$.

- it is geometrically placed in such a way (via functions ψ_1 and ψ_2) so that the computation of $\mathbb{Z}_2$-index is technically easier afterwards.

This space is called $Join(B^d)$. To achieve both these goals, the exact embedding is as follows. Consider the case when $d = 1$: a point $x = (x_1), x_1 \in \mathbb{R}$ in the first copy of B^1 goes to the point $(1, x_1, 0, 0) \in \mathbb{R}^4$. And the point $y = (y_1), y_1 \in \mathbb{R}$ in the second copy of B^1 goes to the point $(0, 0, 1, y_1) \in \mathbb{R}^4$.

Then each point in the line-segments between them is of the form

$$t \cdot (1, x_1, 0, 0) + (1 - t) \cdot (0, 0, 1, y_1) = (t, tx_1, 1 - t, (1 - t)y_1)$$

It is easy to see that there is a unique way to write each point in $Join(B^d)$: given any point, the value of t is fixed by the first and third coordinates, and once t is fixed, x and y values are fixed. Therefore no two line-segments intersect.

For an arbitrary value of d: the point $x \in \mathbb{R}^d$ in the first copy of B^d goes to the point $(1, x_1, \ldots, x_d, 0, \ldots, 0) \in \mathbb{R}^{2d+2}$ (note: there are $(d + 1)$ zeroes at the end). And the point $y \in \mathbb{R}^d$ in the second copy of B^d goes to the point $(0, \ldots, 0, 1, y_1, \ldots, y_d) \in \mathbb{R}^{2d+2}$ (note: there are $(d + 1)$ zeroes at the start). Again it can be verified that no two line-segments intersect.

Finally, we delete the middle point of each line segment between the same point x in the first copy and it's mirror point x in the second copy; i.e., delete the point $\frac{1}{2} \cdot \psi_1(x) + \frac{1}{2} \cdot \psi_2(x)$, for all x in B^d.

The remaining set of points is the final space Y.

It remains to give a $\mathbb{Z}_2$-structure to Y. That is as follows:

$(x, y, t) \Longleftrightarrow (y, x, 1 - t)$. Equivalently, $t \cdot \psi_1(x) + (1 - t) \cdot \psi_2(y) \Longleftrightarrow (1 - t) \cdot \psi_1(y) + t \cdot \psi_2(x)$

So the point lying on the line-segment defined by the point $\psi_1(x)$ (i.e., from the first copy of B^d) and the point $\psi_2(y)$ (the second copy of B^d), and with parameter t is assigned the antipodal point lying on the line-segment defined by the point $\psi_1(y)$ (the first copy of B^d), the point $\psi_2(x)$ (the second copy of B^d), and with parameter $(1 - t)$.

4. $Ind_{\mathbb{Z}_2}(Y)$

Given our fixed embedding of Y, this is now easy: we will observe that Y is now just a ($\mathbb{Z}_2$-preserving) subset of the deleted product $(\mathbb{R}^{d+1})^2_\Delta$! As $Ind_{\mathbb{Z}_2}((\mathbb{R}^{d+1})^2_\Delta) \leq d$, we get the same upper-bound on $Ind_{\mathbb{Z}_2}(Y)$.

First consider the case $d = 1$. To see $\mathbb{Z}_2$-preservation, a point in Y represented by (x, y, t), say $x = (x_1), y = (y_1)$, has coordinates $(t, tx_1, 1 - t, (1 - t)y_1)$. And it's antipodal point, represented by $(y, x, 1 - t)$, has coordinates $(1 - t, (1 - t)y_1, t, tx_1)$. But this is the same as the $\mathbb{Z}_2$ antipodality defined on $(\mathbb{R}^{d+1})^2_\Delta$, where a point with coordinate (x_1, x_2, x_3, x_4) has antipodal point (x_3, x_4, x_2, x_1). So the antipodal (under $\mathbb{Z}_2$-action on $(\mathbb{R}^{d+1})^2_\Delta$) of the point with coordinates $(t, tx_1, 1-t, (1-t)y_1)$ is the point with coordinates $(1-t, (1-t)y_1, t, tx_1)$.

Finally, note that no point of Y (which are all of the form $(t, tx_1, 1-t, (1-t)y_1)$) maps to the (deleted) diagonal (x_1, x_2, x_1, x_2), as that would imply $t = 1/2$, and $x_1 = y_1$; but we deleted all points of this type from Y.

Exactly same argument works for general d. Therefore $Ind_{\mathbb{Z}_2}(Y) \leq d$.

2. Space X

We will construct X in a very similar manner to Y. Let $k = d+1$. Take two copies of Δ^k, and embed them in a high-enough $\mathbb{R}^n$ such that the line-segments from a point from one copy to the point in the other copy are disjoint. Again, each point can be represented by the tuple (x, y, t). Then delete all line-segments between x in the first copy, and y in the second copy, where x and y have intersecting support.

This is the space X.

The antipodal action on X is the same as that of Y:

$$(x, y, t) \Longleftrightarrow (y, x, 1-t)$$

where x is a point in the first copy of Δ^k, and y is a point in the second copy of Δ^k.

3. $Ind_{\mathbb{Z}_2}(X)$

We show that X is homeomorphic (with a $\mathbb{Z}_2$-map) to the crosspolytope β_k, where $k = d+1$.

X was made up of points represented by tuples of type (x, y, t), where x lies in the first copy of Δ^k and y in the second copy of Δ^k. Label the vertices of the first copy of Δ^k as $p_0, \ldots, p_k$, and the vertices of the second copy as $q_0, \ldots, q_k$, where p_i and q_i are copies of the same vertex of Δ^k.

Consider the following map $g : X \to \beta_k$. $g(p_i) = e_i$, and $g(q_i) = -e_i$. The rest of the map is defined by linear extension, i.e., consider a point of X represented by (x, y, t). Let $x = \sum_{i \in I} \lambda_i p_i$, $y = \sum_{j \in J} \mu_j q_j$. By our construction of X, we have $I \cap J = \emptyset$. Then the point denoted by (x, y, t) is the point:

$$t \cdot x + (1-t) \cdot y = t \cdot (\sum_{i \in I} \lambda_i p_i) + (1-t) \cdot (\sum_{j \in J} \mu_j q_j), \text{ where } \sum \lambda_i = 1, \sum \mu_i = 1$$

and so g maps it to:

$$\begin{aligned} g\left(t \cdot (\sum_{i \in I} \lambda_i p_i) + (1-t) \cdot (\sum_{j \in J} \mu_j q_j) \right) &\to t \cdot (\sum_{i \in I} \lambda_i g(p_i)) + (1-t) \cdot (\sum_{j \in J} \mu_j g(q_j)) \\ &= t \cdot (\sum_{i \in I} \lambda_i e_i) + (1-t) \cdot (\sum_{j \in J} \mu_j (-e_j)) \end{aligned}$$

We claim that g is a homeomorphism between X and β_k. Furthermore, it is also a $\mathbb{Z}_2$-map; here we use our constructed $\mathbb{Z}_2$-structure on X, and the standard $x \leftrightarrow -x$ antipodality on β_k.

First note that indeed each point is mapped to β_k, as the coefficients sum up to one, i.e., $\sum t\lambda_i + \sum(1-t)\mu_i = 1$. And as $I \cap J = \emptyset$, the point does lie on a face of β_k.

Now take any point in β_k, say the point $b = \sum_{i\in I} \lambda_i e_i + \sum_{j\in J} \mu_j(-e_j)$, where $\sum_i \lambda_i + \sum_j \mu_j = 1$, and $I \cap J = \emptyset$. Then let $t = \lambda = \sum \lambda_i, \mu = \sum \mu_j = 1-t$, and consider the point

$$x = t(\sum_{i\in I} \frac{\lambda_i}{\lambda} g^{-1}(e_i)) + (1-t)(\sum_{j\in J} \frac{\mu_j}{\mu} g^{-1}(-e_j)) = t(\sum_{i\in I} \frac{\lambda_i}{\lambda} p_i) + (1-t)(\sum_{j\in J} \frac{\mu_j}{\mu} q_j)$$

As $\sum_i \lambda_i/\lambda = 1, \sum_j \mu_j/\mu = 1$, and $I \cap J = \emptyset$, x lies in X, and clearly $g(x) = b$.

Finally, it has to be shown that g is a $\mathbb{Z}_2$-map. Therefore,

$$(x, y, t) \iff (y, x, 1-t)$$

or in geometric coordinates

$$t \cdot (\sum_{i\in I} \lambda_i p_i) + (1-t) \cdot (\sum_{j\in J} \mu_j q_j) \iff (1-t) \cdot (\sum_{j\in J} \mu_j p_j) + t \cdot (\sum_{i\in I} \lambda_i q_i)$$

On the other hand, applying g to both the antipodal points:

$$t \cdot (\sum_{i\in I} \lambda_i e_i) + (1-t) \cdot (\sum_{j\in J} \mu_j(-e_j)) \iff (1-t) \cdot (\sum_{j\in J} \mu_j e_j) + t \cdot (\sum_{i\in I} \lambda_i(-e_i))$$

Now β_k is homeomorphic to S^k (with a $\mathbb{Z}_2$-map), and S^k has $\mathbb{Z}_2$-index k. Therefore $Ind_{\mathbb{Z}_2}(X) = k = d + 1$.

3. Map f_{join} from X to Y

This is straightforward:

$$f_{join} : X \to Y \text{ is defined as } f_{join}(\ (x, y, t)\ ) \to (f(x), f(y), t)$$

Note that this maps a copy of Δ^{d+1} to a copy of B^d. It is a $\mathbb{Z}_2$-map, as the point (x, y, t) is mapped to $(f(x), f(y), t)$, while it's antipodal point $(y, x, 1-t)$ is mapped to $(f(y), f(x), 1-t)$.