Universität des Saarlandes
FR 6.2 Informatik

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Exercises for Graph Theory

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss11/graph_theory/

Assignment 1 Deadline: Monday, April 18, 2011

Rules: There are 4 regular exercise problems. The first two serve as preparation for the test conducted in the exercise class, which has problems worth 8 points. You should solve them, therefore, but you do not need to hand them in. The remaining two problems have to be handed in *nicely written up as you would do in a thesis*. Each problem can yield up to 4 points. You need to collect at least 50% of all these points (tests and written homework) from (i) the first three exercise sessions, (ii) the first six sessions, and (iii) the whole term. Occasionally, there might be bonus problems, which yield additional bonus points. They are, typically, more difficult or not that closely related to that week's content of the lecture.

General assumption: Whenever not specified otherwise, $G = (V, E)$ is a graph.

Exercise 1+2 (*oral homework, total 8 points via test*)

Read, learn by heart, and understand all definitions in Chapter 1 up to (and including) Section 1.2 of the Diestel book (both third and fourth edition work; if you use the German version, it will be Chapter 0 up to Section 0.2). Be able to answer questions like the following:

a) Show that if U is an independent set in G, then U is a clique in $\overline{G}$. (A *clique* in G is a subgraph of G which is complete.)

b) True or false: A graph has always more vertices than edges.

c) Write the following statement and its negation with quantifiers: "There is a vertex that is a neighbor of all other vertices but one". Example: If the statement was "G is a complete graph", then the solution would be "$\forall u, v \in V : \{u, v\} \in E$" and for the negation "$\exists u, v \in V : \{u, v\} \notin E$".

d) Give three pair-wise non-isomorphic graphs having three edges and four vertices.

e) Show that for any graph with $|V| \geq 2$ nodes there exist two vertices with the same degree.

f) For $n \geq 2$ specify a graph that has $n - 1$ different vertex degrees.

g) Let $V = \{0, 1\}^d$ and $E := \{\{x, y\} \mid \exists! i \in \{1, \ldots, d\} : x_i \neq y_i\}$. Determine $|V|$, $|E|$, and the degree of each vertex. Note: $\exists!$ means "there is exactly one...".

Exercise 3 (*written homework, 4 points*)

A *directed graph* (possibly with self-loops and parallel edges in opposite direction) is a pair (V,E) with V a set and $E \subseteq V \times V$. The edge $e = (v_1, v_2)$ is said to initiate (start) from v_1 and terminate (end) at v_2.

A *walk* in G is an alternating sequence of vertices and edges $v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1}$ such that $e_i = (v_i, v_{i+1})$. It is called *Eulerian tour*—just as for the undirected case—if $v_1 = v_{k+1}$ and it contains every edge exactly once.

A directed graph G is *connected* if the corresponding undirected graph $G_0 = (V, \{\{u,v\} | (u,v) \in E\})$ is connected, that is, contains a path between any two vertices.

Prove the following characterization: A finite directed graph with no isolated vertices has an Eulerian tour if and only if it is connected and for every vertex v the number of edges terminating in v equals the number of edges starting from v.

Exercise 4 (*written homework, 4 points*)

Determine the length of the shortest $\{0,1\}$-sequence that contains all $\{0,1\}$-sequences of length 8 as contiguous subsequence.
(The sequence $t_1, \ldots, t_m$ has $s_1, \ldots, s_k$ as a contiguous subsequence if $t_{1+j}, \ldots, t_{k+j} = s_1, \ldots, s_k$ for some $j \in \mathbb{N} \cup \{0\}$.)

[Hint: use exercise 3.]

Exercise Bonus 1 (*4 Points*)

The academic system of Quasiland has a huge list of n different (non-permanent) academic positions (e.g., PhD student, postDoc, research assistent, junior professor, Privatdozent, ...). A successful academic career consists of being promoted from the current rank to the next rank until being promoted from the nth rank, in which case one receives a permanent professor position.

Promotion, unfortunately, is non-trivial. It works as follows. Each year, the dean of the school suggests to the ministry of education that some part of the non-permanent staff is promoted. If the ministry accepts this proposal, all these people will be promoted and all the rest will be fired (but get well paid jobs in the flourishing Quasiland industry). If the ministry disagrees with the proposal, all proposed people will go to industry and the non-proposed will be promoted. There is no staying on the current rank (“up or out”).

Obviously, the two players dean and ministry have opposing aims. The dean is happy to gain another professor helping with teaching, the ministry tries to prevent this and safe the professor’s salary.

For a general situation with x_i people initially on rank i, $i = 1, \ldots, n$, find sufficient and necessary conditions for the dean to be able to push someone to a professor’s position. Example: If $n = 4$ and $x_1 = 3$, $x_2 = 1$, $x_3 = x_4 = 0$, then the dean won’t succeed.