Universität des Saarlandes
FR 6.2 Informatik

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Exercises for Graph Theory

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss11/graph_theory/

Assignment 4 Deadline: Thursday, May 12, 2011

Rules: The first two problems serve as preparation for the test conducted in the exercise class. You should solve them, therefore, but you do not need to hand them in. The test yields 8 points. The remaining problems have to be handed in *nicely written up as you would do in a thesis, as the Diestel does, ...* in the Thursday lecture. These homework problems yield 8 points in total.

You need to collect at least 50% of all these points (tests and written homework) from (i) the first three exercise sessions, (ii) the first six sessions, and (iii) the whole term.

Occasionally, there might be bonus problems, which yield additional bonus points. They are, typically, more difficult or not that closely related to that week's content of the lecture.

Exercise 1 (*oral homework, in total 8 points via test*)

(a) Read, learn by heart, and understand all definitions in Sections 2 and 2.1. of the Diestel book (except the things related to coverings and packings, or to Theorem 2.1.4).

(b) Draw a graph on six vertices that has a minimum-cardinality vertex cover of size 3.

(c) Draw a *connected non-bipartite* graph on six vertices that has a minimum-cardinality vertex cover of size 3.

(d) What is the size of a minimum-cardinality vertex cover of the complete graph on n vertices?

(e) What is the size of a maximum-cardinality matching of the complete graph on n vertices?

(f) Give an example of a matching in a graph that is maximal w.r.t. subgraph inclusion but not maximum-cardinality. Try to make your example as small as possible.

Exercise 2 (*oral homework, in total 8 points via test*)

Read and fully understand the *first* proof of Hall's Theorem (Theorem 2.1.2) in the Diestel book.

Exercise 3 (*written homework, 3 points*)

Prove the following lemma that was stated in the lecture:

Lemma. Let $G = (V,E)$ be a graph, and let $M \subseteq E$ be a matching. M is a maximum-cardinality matching if and only if there is no augmenting path in G w.r.t. M.

Hints: The 'only if' statement (the forward implication) is quite straightforward. For the 'if' part (the backward implication), show that the existence of a matching $M' \subseteq E$ with $|M'| > |M|$ implies the existence of an augmenting path in G w.r.t. M. To do so, consider the symmetric difference $M' \triangle M := (M' \setminus M) \cup (M \setminus M')$.

Exercise 4 (*written homework, 2 points*)

Let G be a bipartite graph with partition classes A and B. Show: If there exists an integer $d \geq 0$ such that $N(S) \geq |S| - d$ for every set $S \subseteq A$, then there exists a matching that matches at least $|A| - d$ vertices of A.

Exercise 5 (*written homework, 3 points*)

An $n \times n$ *Latin square* is an $n \times n$ matrix with entries in $\{1, \ldots, n\}$ such that no row contains a number twice and no column contains a number twice. If the first r rows of the matrix have been filled with integers in $\{1, \ldots, n\}$ such that these conditions hold, we speak of an $r \times n$ *Latin square*.

Show that any $r \times n$ Latin square can be completed to an $n \times n$ Latin square by filling in the remaining rows.

Hint: Proceed row by row, and apply a result from the lecture to to an appropriately defined bipartite graph.

Exercise Bonus 1 (*4 Points*)

Moving alternately, two players jointly construct a path in some fixed graph G. In the very first move of the game, the first player chooses a vertex of G as the starting point v_1 of the path. In all subsequent moves, if $v_1 \ldots v_n$ is the path constructed so far, the player to move next has to find a vertex v_{n+1} such that $v_1 \ldots v_{n+1}$ is again a path. Whichever player cannot move loses. For which graphs G does the first player have a winning strategy, for which the second?