Universität des Saarlandes
FR 6.2 Informatik

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Exercises for Graph Theory

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss11/graph_theory/

Assignment 12 Deadline: Thursday, July 7, 2011

Rules: The first problem serves as a preparation for the test conducted in the exercise class. You should solve it, but you do not need to hand it in. The test yields 8 points. The remaining problems have to be handed in *nicely written up as you would do in a thesis, as the Diestel does, ...* in the Thursday lecture. These homework problems yield 8 points in total.

You need to collect at least 50% of all these points (tests and written homework) from (i) the first three exercise sessions, (ii) the first six sessions, and (iii) the whole term.

Occasionally, there might be bonus problems, which yield additional bonus points. They are, typically, more difficult or not that closely related to that week's content of the lecture.

Exercise 1 (*oral homework, in total 8 points via test*)

Read and understand the material in Chapter 11 up to and including Theorem 11.1.3, and the material in Chapter 9 up to but without Theorem 9.1.2.

Also, read and understand the sample solution for the bonus exercise of Assignment 9. Note that the solution for part b) is an alternative proof of Theorem 9.1.1 (or, rather, the same proof in a different guise).

Exercise 2 (*written homework, 2 Points*)

Recall that by G_n we denote a graph drawn uniformly at random from $\mathcal{G}_n$, the set of all graphs on $V_n = \{0, \dots, n-1\}$.

a) Determine the probability that G_n is isomorphic to
   i) the complete graph K_n,
   ii) the graph on n vertices that has exactly one edge,
   iii) C_n, the cycle of length n.

b) Answer the same questions for $G_{n,p}$, the random graph on V_n obtained by including each of the $\binom{n}{2}$ possible edges with probability p independently.

Exercise 3 (*written homework, 2 Points*)

Recall that we say that a graph property $\mathcal{P}$ *holds for almost all graphs* if and only if $\lim_{n\to\infty} \Pr[G_n \text{ has } \mathcal{P}] = 1$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two graph properties that hold for almost all graphs. Prove or disprove that the following properties also hold for almost all graphs.

i) $\mathcal{C}$:= 'The graph has property $\mathcal{A}$ *or* property $\mathcal{B}$'

ii) $\mathcal{D}$:= 'The graph has property $\mathcal{A}$ *and* property $\mathcal{B}$'

Exercise 4 (*written homework, 1.5 Points*)

Determine the exact probability (as a function of p) that two independently drawn random graphs $G_{3,p}$ (these are random graphs on 3 vertices) are isomorphic. [Sanity check: what should you get for $p = 0$ or $p = 1$?]

Exercise 5 (*written homework, 2.5 Points*)

a) Use Ramsey's Theorem (Theorem 9.1.1) to show that for every integer $k \geq 2$ there exists $n \in \mathbb{N}$ such that every sequence of n distinct integers contains an increasing subsequence of length k or a decreasing subsequence of length k. [1.5P.]

b) Find an example showing that n must be larger than $(k-1)^2$. [1P.]

Exercise Bonus 1 (*4 Points*)

Prove that almost every graph contains a triangle; i.e., prove that

$$\lim_{n\to\infty} \Pr[G_n \text{ contains a triangle}] = 1.$$

(This also follows from a more general result we will discuss in the lecture. You are not allowed to use this result, nor to mimick its proof. More specifically, you are not allowed to use the Second Moment Method for this exercise.)