Universität des Saarlandes
FR 6.2 Informatik

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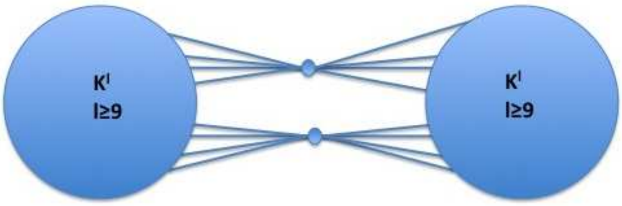
Solution for Exercise 3

Exercise 1 (*oral homework, in total 8 points via test*)

(a) Read, learn by heart, and understand all definitions in Sections 1.4 through 1.6 of the Diestel book.

(b) Draw a graph G that has edge-connectivity $\lambda(G) = 8$ and (vertex-)connectivity $\kappa(G) = 2$.

Solution:



where K^l denotes the complete graph on l vertices.

(c) Let G have at least 2 vertices. What do you need to show in order to prove that $\lambda(G) \leq 25$? Do not use the word 'ℓ-edge-connected' for $\ell \geq 2$ in your answer.

Solution: There is a set of at most 25 edges in $E(G)$ whose removal disconnects G.

(d) Spell out what it means for a graph G *not* to be maximally acyclic (see (iv) in Exercise 3 below).

Solution: Either G is cyclic or there is an edge $xy \notin E(G)$ such that $G + xy$ is acyclic.

(e) Prove, using the classic 'consider a longest path' argument, that every tree has a leaf.

Solution: Let T be an arbitrary tree. Consider a longest path $P := v_1v_2...v_k$ in T. We show that v_k is a leaf in T. Assume v_k is not a leaf in T, then it has a neighbor $u \neq v_{k-1}$. Since P is maximal (i.e. cannot be extended) we know u lies on the path P. Hence, T has a cycle. This contradicts with the fact that it is a tree.

(f) Read and understand the proof of Corollary 1.5.3 in the Diestel book.

(g) Let T be a tree. Is it true that every graph G that satisfies $\delta(G) \geq |V(T)| - 1$ and has girth at least $|V(T)| + 1$ contains an *induced* copy of T?

Solution: Yes. From Corollary 1.5.4 from Diestel we know there is a subgraph $T' \subseteq G$ that is isomorphic to T. For the sake of contradiction assume T' is not induced, i.e., there exists an edge e in G that connects two vertices that are in T' and e is not in T'. By Theorem 1.5.5, T' is maximally acyclic so $T' + e \subseteq G$ contains a cycle of length less than or equal $|V(T)|$. This contradicts with the assumption that G has girth at least $|V(T)| + 1$.

(h) True or false: Each 4-partite graph is 5-partite.

Solution: It depends on whether we allow partition classes to be empty (Diestel doesn't but many other authors do in this context). If we allow empty partition classes then the statement is true (since we can add an empty partition class). Otherwise it is false (consider K^4, the complete graph on 4 vertices).

Exercise 2 (*oral homework, in total 8 points via test*)

Read and fully understand the proof of Theorem 1.4.3 in the Diestel book.

Exercise 3 (*written homework, 4 points*)

Prove Theorem 1.5.1 in the Diestel book. I.e., show that the following assertions are equivalent for a graph T:

(i) T is a tree.

(ii) Any two vertices of T are linked by a unique path in T.

(iii) T is a minimally connected, i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$.

(iv) T is maximally acyclic, i.e. T contains no cycle but $T + xy$ does, for any two non-adjacent vertices $x, y \in T$.

Solution:

i) $\Rightarrow$ *ii)*: A tree T is an acyclic connected graph. By connectivity, there is at least one path linking each two vertices in T. For the sake of contradiction, assume there are two different paths P_1 and P_2 from vertex u to vertex v in T. Let a be the first vertex after which P_1 and P_2 differ (a exists since both paths start at u and are different). Let b be the first vertex after a in P_1 that is also contained in P_2 (b exists since v is such a vertex). Then, aP_1bP_2a is a cycle in T (because of the choice of b no vertex appears twice on this cycle), contradicting the assumption that T is a tree.

ii) $\Rightarrow$ *iii)*: Let T be an arbitrary graph in which any two vertices are linked by a unique path. Observe that T is connected. Suppose there exists an edge $e = uv$ that can be removed such that $T - e$ is not disconnected. Then there is a path P from u to v in $T - e$. Then $P + e$ will contain two different paths between u and v, which contradicts the hypothesis.

iii) $\Rightarrow$ *iv)*: Let T be an arbitrary minimally connected graph. We first show that T is acyclic. Suppose T is not acyclic, and $C = v_0v_1...v_kv_0$ is a cycle in T. Then the graph $T - v_0v_1$ is still connected since any path in T can be transformed into a walk in $T - v_0v_1$ by replacing any occurrence of the edge v_0v_1 by the path $C = v_kv_{k-1}\dots v_1$. Assume now that $T + v_0v_1$ is acyclic where v_0v_1 is not an edge in T. Then T is not connected, because any path $v_0u_1 \dots u_kv_1$ in T would give a cycle $v_0u_1 \dots v_1u_kv_0$ in $T + v_0v_1$. This contradicts the hypothesis.

iv) $\Rightarrow$ *i)*: Let T be an arbitrary maximally acyclic graph. It is obvious that T is acyclic, so we only need to show that T is connected. Suppose it is not connected. Then there exist two vertices u and v in T that are not connected by a path. In particular $uv \notin T$. However, then the graph $T + uv$ is acyclic since there is no other path from u to v that can close the cycle. This is a contradiction to T being maximally acyclic.

Exercise 4 (*written homework, 4 points*)

(a) Show that in a 2-connected graph every vertex is contained in a cycle. [2P.]

Solution: Let G be a 2-connected graph. Let v be an arbitrary vertex of G. Using Prop. 1.4.2 from the Diestel book we know that G is 2-edge-connected and that v has at least two neighbors, say u and w. Due to 2-edge-connectivity, we know $G - uv$ is connected. Hence there exists a path from u to v in $G - uv$, and Pvu is a cycle in G containing v.

(b) Let G be a graph. Show that G and its complement $\overline{G}$ cannot have both diameter larger than 3. [2P.]

Solution: If the graph G has diameter at most 3 then we are done.
Suppose G has a diameter greater than 3. We have to prove that $\overline{G}$ has diameter at most 3, i.e., that any two vertices are connected by a path of length at most 3 in $\overline{G}$. If G is disconnected then $\text{diam}(\overline{G}) \leq 3$ will hold, because in $\overline{G}$ all vertices from G that lie in different components will be connected by an edge, and each 2 vertices in the same component will be connected to a common vertex from a different component, inducing a path of length 2. (Thus in this case, we even have $\text{diam}(\overline{G}) \leq 2$.)

If G is connected with diameter greater than 3, then there is a path P of length greater than 3 in G. Call its endpoints u and v. The graph $\overline{G}$ will surely contain the edge uv. In $\overline{G}$ any vertex different from u and v is adjacent to u or to v because there is no path of length 2 connecting u and v in G. Let x and y be any two vertices different from u and v. If they share u or v as a common neighbor in $\overline{G}$, then xuy or xvy is a path connecting them in $\overline{G}$. Otherwise, $xvuy$ or $xuvy$ is a path connecting them in $\overline{G}$. In any case x and y have distance at most 3. Our argument also shows that every vertex x has distance at most 2 from u and v.