Universität des Saarlandes
FR 6.2 Informatik

Prof. Dr. Benjamin Doerr, Dr. Danny Hermelin, Dr. Reto Spöhel **Summer 2011**

Solution for Exercise 8

General assumption:
Unless stated otherwise, we always color any graph G with numbers $\{1, \ldots, \chi(G)\}$.

Exercise 1

(a) Read carefully and understand all material in Section 4.4 of the Diestel book.

(b) Why is every maximal planar graph with at least four vertices 3-connected?

Solution: By Kuratowski's Theorem (Theorem 4.4.6) we know that a graph is planar iff it contains neither $K_{3,3}$ nor K_5 as a topological minor. Knowing this, it suffices to show that every maximal graph with at least four vertices that does not contain neither $K_{3,3}$ nor K_5 as a topological minor is 3-connected. This is precisely Lemma 4.4.5.

(c) Read carefully and understand the material in Section 5.0 and 5.2 (up to 5.2.2.) of the Diestel book.

(d) Explain why Corollary 5.2.3 in the Diestel book follows.

Solution: We know that

$$\chi(G) \leqslant \mathrm{col}(G) = \max\{\delta(H) : H \subseteq G\} + 1.$$

Let $\overline{H}$ be a subgraph that achieves the maximum in the formula above. Hence, $\overline{H}$ is a subgraph of G with minimum degree

$$\mathrm{col}(G) - 1 \geqslant \chi(G) - 1.$$

(e) Give an example of a graph whose chromatic number equals its chromatic index.

Solution 1: For any even cycle C_{2k}, $\chi = \chi' = 2$.
Solution 2: For any odd cycle C_{2k+1}, $\chi = \chi' = 3$.

(f) Give an example of a graph whose chromatic number is strictly higher than its chromatic index.

Solution: For an isolated edge, $\chi = 2 > \chi' = 1$.

(g) Give an example of a graph whose coloring number equals its chromatic number.

Solution: For any odd cycle C_{2k+1}, $\chi = \text{col} = 3$.

(h) Give an example of a graph whose coloring number is strictly higher than its chromatic number.

Solution 1: For any even cycle C_{2k}, $\text{col} = 3 > \chi = 2$.
Solution 2: For $K_{2,2,2}$, $\text{col} = 5 > \chi = 3$.

(i) Determine the chromatic index of the complete graph on 4 vertices.

Solution: Clearly, $\chi' \geqslant 3$ as the graph is 3-regular. It's easy to see that each of the three pairs of non-incident edges can use one color, so $\chi' = 3$.

(j) Determine the chromatic index of the complete graph on 5 vertices.

Solution: It is easy to check that no color can be used more than twice, so $\chi' \geqslant |E|/2 = 5$. It is indeed easy to find a proper edge coloring with 5 colors, and so $\chi' = 5$.

Exercise 2 *(written homework, 2 points)* Let G_1 and G_2 be two graphs with $|V(G_1) \cap V(G_2)| < k$, and let X be a k-connected graph. Prove that if $G := G_1 \cup G_2$ contains a subdivision H of X, then all branch vertices of H must belong to either $V(G_1)$ or $V(G_2)$. (Note that this is a generalization of the observation we used throughout the proof of Lemma 4.4.4 in the lecture.)

Solution:
Proof. For the sake of contradiction, let $a \in V(G_1) \setminus V(G_2)$ and $b \in V(G_2) \setminus V(G_1)$ be branch vertices of H in G. Since X is k-connected, the vertices of X corresponding to a and b are connected by k independent paths in X (by Menger's Theorem). These give rise to k independent $a{-}b$ paths in H. Each of those paths uses at least one vertex from $V(G_1) \cap V(G_2)$. However, as $|V(G_1) \cap V(G_2)| < k$, there are no k independent such paths. This is the desired contradiction, and proves the claim. □

Exercise 3 *(written homework, 2 points)* Show that the chromatic number of a graph G is exactly the maximum of the chromatic numbers of the blocks of G.

Solution:
Proof. Clearly, $\chi(G)$ colors are sufficient to properly color all blocks of G, all we need to show is how to properly color G having a proper coloring of its blocks. W.l.o.g. let G be connected. We order the blocks of the graph as follows: we take an arbitrary block as B_0, and as B_i ($i > 0$) we take any block that has a vertex in $\bigcup_{k<i} B_k$. Since G is connected, such a block always exists; and since block graphs are forests, B_i has exactly one vertex in common with $\bigcup_{k<i} B_k$. Let us prepare for every block B_i a proper coloring with colors $\{1, \ldots, \chi(B_i)\}$; we will refer to this as its *intended coloring*. We color the graph in the order established above. We color B_0 with its intended coloring. For any subsequent block B_i, the color of exactly one vertex is already set by our coloring of one of the previous blocks. We compare this color with the color of the same vertex in the intended coloring for B_i. If the colors match, we apply the intended coloring; if they are different we apply the intended coloring with those two colors swapped. After the swapping, we still have a proper coloring of B_i, and in fact a proper coloring of $\bigcup_{k\leq i} B_k$. As swapping does not increase the total number

of colors used, the claim follows by induction. (Note that the final coloring may use a color higher than $\chi(B_i)$ in some block B_i, as our procedure may swap in a color that was not used in the intended coloring of B_i.) $\square$

Exercise 4 *(written homework, 4 points)*

a) Show that every finite graph G has a vertex-ordering for which greedy coloring uses exactly $\chi(G)$ colors. [1P.]

   Solution:
   Proof. Let us enumerate the vertices in the graph according to their increasing color number in some fixed proper coloring c that uses exactly $\chi(G)$ colors. It is easy to check by induction that greedy coloring will use for every vertex v color $c(v)$ or lower. $\square$

b) Explain why what you showed in (a) does *not* imply that $\chi(G) = \mathrm{col}(G)$ for every graph G, and give an example of a graph G with $\chi(G) < \mathrm{col}(G)$. [1P.]

   Solution:
   The coloring number gives a pessimistic estimator on the behavior of the greedy algorithm. Namely, if a vertex has a degree $d-1$, we *might* need d colors in order to be able to color it. Greedy may actually use a lower color if some neighbors of the vertex have the same color, so it may achieve results better than the coloring number. $K_{k,k}$ is k-regular, so it has coloring number $k+1$, but it is bipartite so it has chromatic number 2.

c) Show that for every integer k there is a graph G with $\chi(G) = 2$ and a vertex-ordering for which greedy coloring uses k colors. [2P.]

   Solution:
   Proof. Let us consider the graph $G := \{\{v_1, \ldots, v_k, w_1, \ldots, w_k\}, \{v_i w_j : i \neq j\}\}$ with its vertices enumerated as $v_1, w_1, v_2, w_2, \ldots, v_k, w_k$. It is easy to see that greedy coloring will use color i exactly for the vertices v_i and w_i, $i = 1, \ldots, k$. Thus it uses k colors in total. However, 2 colors would be sufficient, as G is bipartite. $\square$