Universität des Saarlandes
FR 6.2 Informatik

Prof. Dr. Benjamin Doerr, Dr. Danny Hermelin, Dr. Reto Spöhel **Summer 2011**

Solution for Exercise 9

Note: Unless explicitly stated otherwise, whenever we speak of a colouring, we mean a *proper* colouring as defined in the lecture.

Exercise 2

(a) We know from Corollary 2.1.3 that d-regular bipartite graphs G have a 1-factor $M \subseteq E(G)$. Clearly $G \backslash M$ is still bipartite and $(d-1)$-regular. Therefore we can inductively partition the edge set of G into a series of d disjoint 1-factors $M_1, \ldots, M_d$. We can now assign the colour i to every $e \in M_i$, for all $1 \leq i \leq d$. This is a valid colouring of the edges, as by the definition of a 1-factor no two edges from the same M_i are incident to a common node. Hence $\chi'(G) \leq d$.

Since G contains a node that is incident to d edges, there can be no edge colouring that uses less colours, therefore $\chi'(G) = d$.

(b) We show that it is possible to augment any bipartite $G = (A \dot\cup B, E)$ with $\Delta := \Delta(G)$ to a Δ-regular bipartite graph. If w.l.o.g. $|A| < |B|$, we add $|B| - |A|$ new nodes to A to obtain $G^{(0)} = (A' \dot\cup B, E)$. Let $H = K_{\Delta,\Delta} - \{x, y\}$ be a complete bipartite graph with parts of size Δ from which one (arbitrary) edge is removed. As long as there is a pair of vertices $u \in A'$, $v \in B$ in $G^{(i)}$ with $\deg(u), \deg(v) < \Delta$, we construct $G^{(i+1)}$ by adding a disjoint copy of H and connecting the endpoints x and y of the missing edge in H to u and v, respectively. This increases $\deg(u)$ and $\deg(v)$ by one, leaves the degrees of all other vertices in $G^{(i)}$ unchanged, and adds 2Δ new vertices of degree Δ. Furthermore $G^{(i+1)}$ is bipartite by construction.

As $|A'| = |B|$ we will run out of vertices $u \in A'$ and $v \in B$ with degree less than Δ at the same time (after exactly $\Delta|B| - E(G)$ steps, to be precise). Therefore, the final graph in this sequence is Δ-regular. Let $\hat{G}$ denote this last graph. By (a) the chromatic index of $\hat{G}$ is Δ and, as G is a subgraph of $\hat{G}$, we have $\chi'(G) \leq \chi'(\hat{G}) = \Delta$. On the other hand, Δ colours are clearly necessary to colour G, as there is at least one vertex of degree Δ in G. Thus $\chi'(G) = \Delta$.

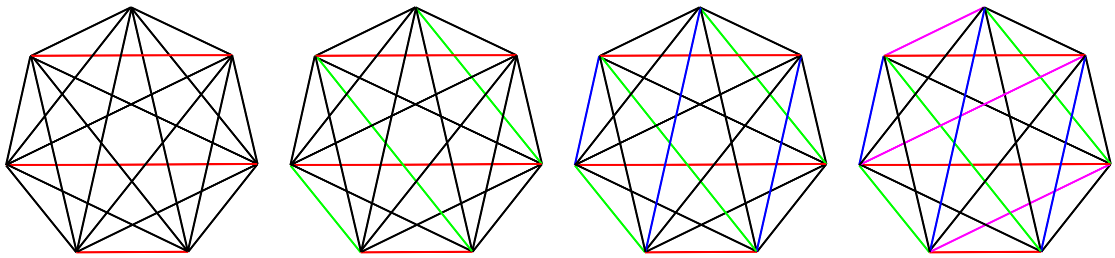
Exercise 3

We claim that

$$\chi'(K_\ell) = \begin{cases} \ell & \ell \text{ odd} \\ \ell - 1 & \ell \text{ even.} \end{cases}$$

First consider the odd case. Let the vertices be numbered from 0 to $\ell - 1$. We construct the colour classes as a series of maximal matchings in G. We construct ℓ matchings by selecting the edges $M_i = \{\{i+k, i-k\} | k = 1, \ldots, (\ell-1)/2\}$, for $i = 0, \ldots, \ell - 1$ where addition is done modulo ℓ. Every node i is contained in $\ell - 1$ matchings (all except M_i), so we cover all edges. By assigning one colour to each matching, we obtain an ℓ-colouring for G. This is clearly optimal, as no colour class can contain more than $(\ell-1)/2$ edges.

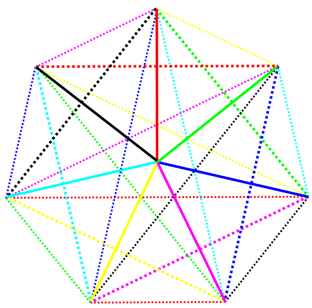
With some imagination this scheme is equivalent to colouring the edges by their slope in a particular drawing of the graph.



The edges are coloured by their slope.

In the even case we fix an arbitrary vertex v and remove it from the graph. The remaining graph has an odd number of vertices and can hence be coloured with $\ell - 1$ colours as described above. In this colouring every colour is missing from exactly one vertex. Let the colour c_i be missing from u_i. Then we construct a colouring for the whole graph with the same number of colours by assigning c_i to the edge $\{v, u_i\}$.

Again with some imagination this can be seen as putting the additional vertex in the middle of the circle, colouring the remaining graph by slope and then colouring every edge to the central vertex with the colour of the matching to which it is orthogonal.



Edges to the central vertex get the colour of the matching to which they are orthogonal.

Exercise 4

(a) —

(b) All four colours must appear in the neighbourhood of the vertex v. We may have $|N(v)| = 4$ and every colour occurs exactly once, or we may have $|N(v)| = 5$, and there is one colour that appears twice.

(c) Consider as in the original proof a cyclic (say clockwise) enumeration $v_1, \dots, v_5$ (or $v_1, \dots, v_4$ if $|N(v)| = 4$) of the neighbours of v in a planar drawing of G, where $c(v_i) = i$, $i = 1, \dots, 4$. In the original argument we showed that either v_1 and v_3 are in different components of $H_{1,3}$ (the subgraph of G induced by the vertices coloured 1 or 3), or v_2 and v_4 are in different components of $H_{2,4}$ (the subgraph of G induced by the vertices coloured 2 or 4). Then we could either recolour v_1 by swapping the colours 1 and 3 in the corresponding component of $H_{1,3}$, or we could recolour v_2 by swapping the colours 2 and 4 in the corresponding component of $H_{2,4}$. This freed up either colour 1 or colour 2, which we could then assign to v.

This still works for the case with four colours if $|N(v)| = 4$, and it even works if $|N(v)| = 5$ provided the two neighbours of v with the same colour are next to each other: Assume w.l.o.g. that both v_4 and v_5 have colour 4. Then any v_1–v_3 path separates v_2 from both v_4 and v_5, and therefore the swapping indicated above never involves v_5 and frees up either colour 1 or colour 2 as before.

This however does not work in the remaining case where the two neighbours of v with the same colour are *not* next to each other: Assume w.l.o.g. that both v_2 and v_5 have colour 2. Then as before any v_1–v_3 path separates v_2 from v_4 (and v_5), but recolouring v_2 does not free up colour 2 (because it is still used for v_5), and recolouring v_4 instead (note that the original argument is symmetric with respect to the roles of v_2 and v_4) might also fail because v_4 and v_5 might be in the same component of $H_{2,4}$. Then swapping the colours in this component recolours v_4 with colour 2 as desired, but at the same time also recolours v_5 with colour 4 — spoiling the entire argument.

Exercise 5

(a) Consider the case where there are exactly six people $p_1, \dots, p_6$ at the party. Assign each person p_i to vertex v_i in a K_6 and colour the edge $\{v_i, v_j\}$ red if p_i knows p_j, blue otherwise. We want to show that there is a monochromatic triangle in the graph as this corresponds to three persons that know (resp. don't know) each other.

The vertex v_1 has five incident edges, at least three of which must have the same colour, say red. Let v_i, v_j, v_k be the other endpoints of the red edges. If any of the edges $\{v_i, v_j\}$, $\{v_i, v_k\}$, $\{v_j, v_k\}$ is red, we obtain a red triangle with v_1. Otherwise v_i, v_j, v_k form a blue triangle.

(b) We show by induction over $k + \ell$ that there is an integer $P(k, \ell)$ such that at any party with $P(k, \ell)$ people, at least k know each other, or ℓ do not know each other. We frame this as a colouring problem on a complete graph with $P(k, \ell)$ vertices, as in (a). We want to show that there exists a red k-clique or a blue ℓ-clique in this graph. By (a), we can set $P(k, \ell) = 6$ for $k + \ell \leq 3$.

For the induction step, assume that $P(k-1, \ell)$ and $P(k, \ell-1)$ exist. We will prove the claim for $\hat{P} := P(k, \ell) := P(k-1, \ell) + P(k, \ell-1)$. Consider an arbitrary coloring of the complete graph on $\hat{P}$ vertices. Fix a vertex v, and let R and B be the sets of vertices that are connected to v via a red, respectively blue, edge. Clearly, $\hat{P} = |R| + |B| + 1$.

By the definition of R, B, and $\hat{P} = P(k, \ell)$, either $|B| \geq P(k, \ell-1)$ or $|R| \geq P(k-1, \ell)$ (as otherwise $|R| + |B| \leq \hat{P} - 2$, a contradiction). W.l.o.g. let $|B| \geq P(k, \ell-1)$. If the subgraph induced by the vertices of B contains a red clique of size k we are done, otherwise by the induction hypothesis this graph must contain a blue clique of size $\ell - 1$, which together with v forms a blue clique of size ℓ.

Remark: The minimal number we can choose as $P(k, \ell)$ is called the (asymmetric) Ramsey number of k and ℓ and is commonly written $R(k, \ell)$. It is not hard to see that the above inductive argument gives an upper bound of $2^{k+\ell}$ (using $P(k, \ell) = 8 = 2^3$ for $k + \ell \leq 3$ as the induction base). For the symmetric case $k = \ell$ this gives $R(k, k) \leq 4^k$; this is best known up to subexponential terms. The best known lower bound for $R(k, k)$ is $2^{k/2}$, again up to subexponential terms.