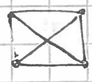
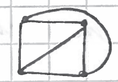
Planar graphs

Def A graph is planar if it can be drawn with open Jordan curves as edges s.t. no 2 edges cross.

 ,  $\rightarrow$ plane graph 

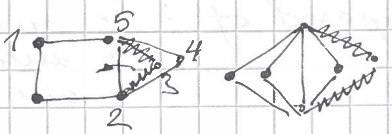
planar;

 – not planar;

Thm: Every planar graph has an embedding on any given point set (assignment of vertices $\rightarrow$ points may be chosen)

Thm: Every planar graph has a straight line embedding

Def. Combinatorial embedding – lists of cyclic orders of edges around each vertex.



$2^{O(n)}$

5: 1 – 4 – 3 – 2
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planar embedding = combinatorial embedding $\oplus$ choice of external face.

Thm (Euler's Formula): $f = \{\#\text{ of faces}\}$

Every non-empty planar graph (connected) satisfies $(\bar{n} - \bar{m}) + f = 2$

$\triangleright$ Induction over $\bar{m}$: $\bar{m} = 0$: $\bar{n} = 1$ $[1 - 0 + 1 = 2]$

$\bar{m} - 1 \rightarrow \bar{m}$: graph G with m edges :

- case 1: G is a tree $[n - (n-1) + 1 = 2]$
- case 2: G is not a tree $\Rightarrow$ G contains a cycle C

T is a spanning tree of G

Take edge $e \in E(G) \setminus E(T)$:

- delete edge e;
- (-1) face f; $\Rightarrow n - (m-1) + f - 1 = 2$ where $(n - m + f = 2)$

In the case of disconnected graphs:

- m increased by $(cc - 1)$ after their connection

$$n - (m + cc - 1) + f = 2 \iff$$

$$n - m + f = 1 + cc \leftarrow \text{For general graphs (non-empty)}$$

Def. A planar graph is <u>maximal planar</u> if adding any edge destroys planarity.

$\Rightarrow$ Every such graph satisfies $m \leq 3n - 6$.

<u>Corollary</u>: All vertices have degree ≥ 6

$2m = \sum \text{degrees}$ thus $2m \geq 6n$ or $m \geq 3n$ – contradict.

$\Rightarrow \exists$ degree at most 5 in a planar graph

$m = O(n) \Rightarrow$ planar graphs are sparse.

<u>Thm</u>: A graph is planar iff it does neither contain a K_5 subdivis. nor a $K_{3,3}$ subdivision.

<u>Thm</u>: A graph is planar iff it does neither contain a K_5 minor nor a $K_{3,3}$ minor.

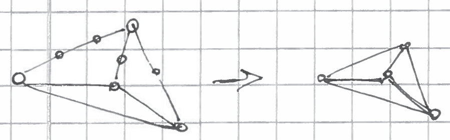
A minor of graph G is a subgraph of a graph, obtained from G by contractions

<u>Lemma</u>: K_5 & $K_{3,3}$-subdivisions are non-planar.

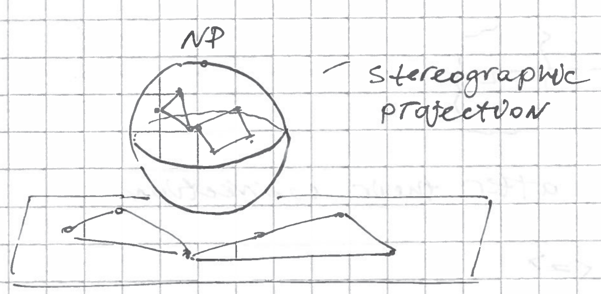
Reduction to K_5, $K_{3,3}$. Thus K_5: $m \leq 3n - 6$ $\rightarrow$ <u>contrad.</u>

$m = 10$, $n = 5$

H- subdivision = K_4-minor



Why "plane" ?



Thm: For every planar embedding and every face f, there is a planar embedding with f as external face.

Detour to Platonic Solids

Def. A plane in $\mathbb{R}^3$ cuts $\mathbb{R}^3$ into two half-spaces.

Def. the intersection of finitely many half-spaces ~~in~~ is called a polyhedron.

Def. A polyhedron is regular if its faces are regular z-gons with the same $\#\ k$ of faces meeting at each vertex

Sum of degrees $= n \cdot k = 2m$

Sum of edges of faces $= z \cdot f = 2m$

$n = \frac{2m}{k}$, $f = \frac{2m}{z}$

$$\frac{2m}{k} - m + \frac{2m}{z} = 2$$

$$\Leftrightarrow \frac{1}{k} + \frac{1}{z} = \frac{1}{m} + \frac{1}{2}$$

thus $k \geq 3, z \geq 3$
$k \leq 5, z \leq 5$

k	z	n	m	f	Name
3	3	4	6	4	Tetrah.
3	4				Cube
					Oktahedron
3	5				Dodekahedron