Dynamic Connectivity

Ran Duan

In this talk

- Concept of dynamic algorithms
- Dynamic connectivity of $O(\log^2 n)$ amortized update time
- Decremental minimum spanning tree

Dynamic Problems

- Find algorithms and data structures to answer a certain query about a set of input objects where each time the input data is modified.

Dynamic Graph

- Fully dynamic model: we can insert and delete edges to the graph G
- Decremental model: only deletions
- Incremental model: only insertions

About dynamic algorithms

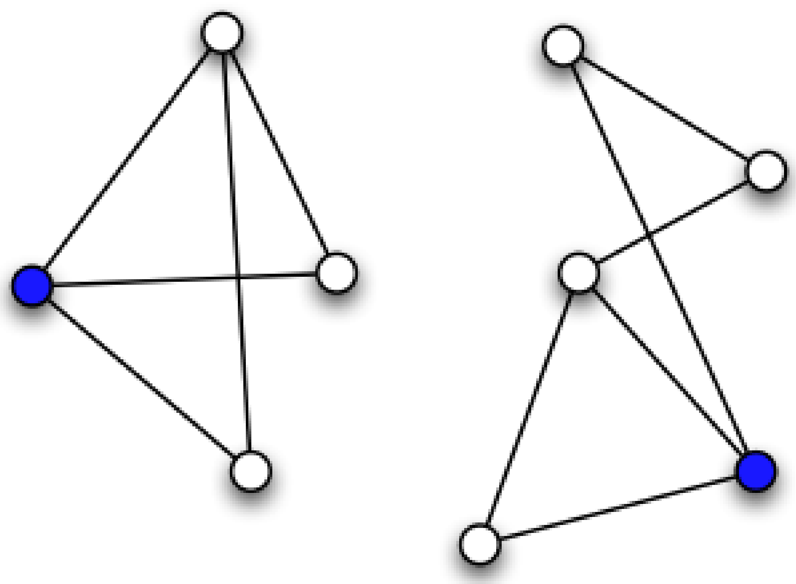
- Measures of complexity:
  - Memory space to store the required data structures
  - Initial construction time for the data structure
  - Insertion/deletion time: time required to modify the data structure
    - Update time
  - Query time: time needed to answer an query

Amortized analysis

- For a sequence of updates, count the average time needed per each update.
  - Some updates may require much longer time
  - Only happen infrequently

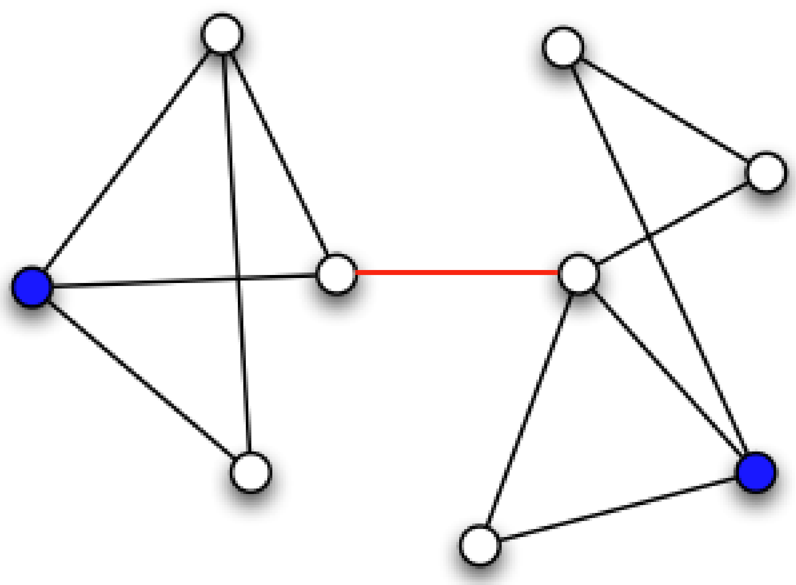
Connectivity Problem

- In an undirected graph G, judge whether any two vertices are connected by a path.



Dynamic Connectivity

- We can insert or delete edges in this graph, and still find the connectivity of any pair of vertices.



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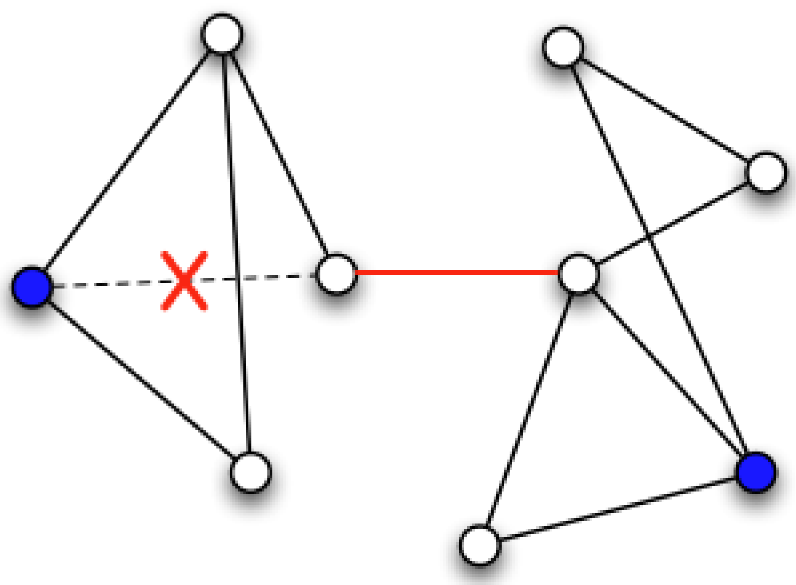
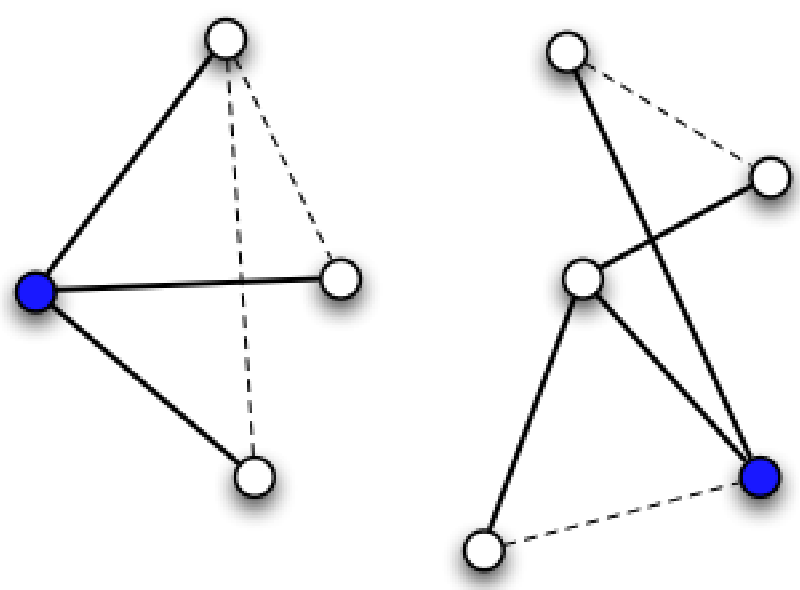


Connectivity and spanning forest

- Spanning forest F: there is a spanning tree in each connected component
- Connectivity: check whether u,v are in the same spanning tree of F.

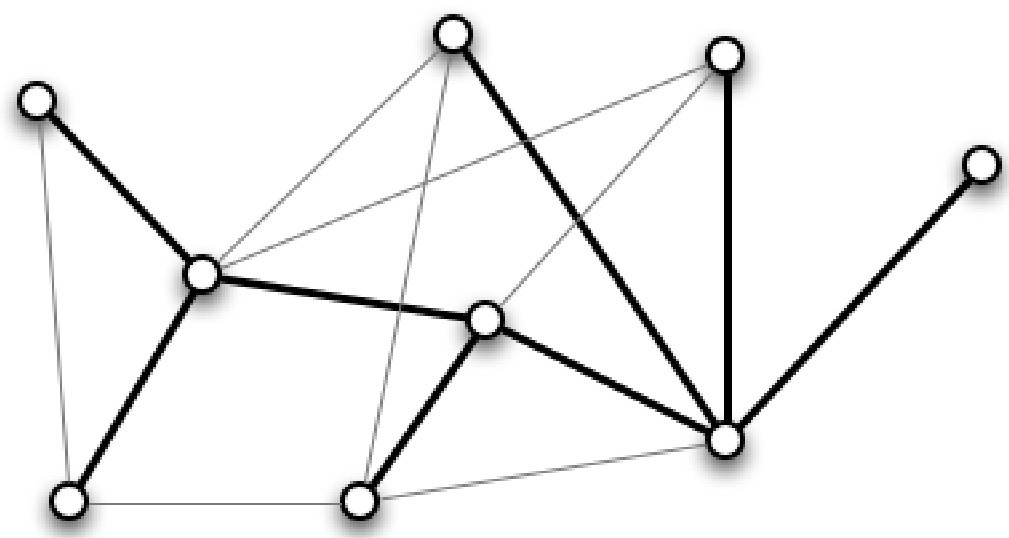


Dynamic Connectivity

- Maintain the spanning forest dynamically
- Inserting (u,v):
  - When u,v are in the same tree, F do not change
  - When u,v are not in the same tree, connect these trees to a bigger tree

Dynamic Connectivity

- Maintain the spanning forest dynamically
- Deleting a tree edge (u,v):
  - The tree will be split into two parts
  - We need to find other edges reconnecting these two parts



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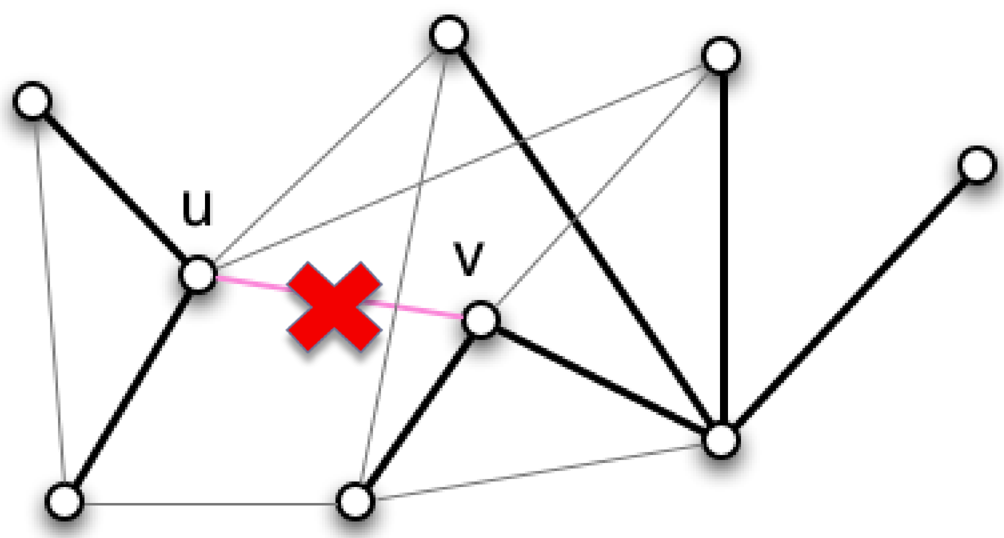

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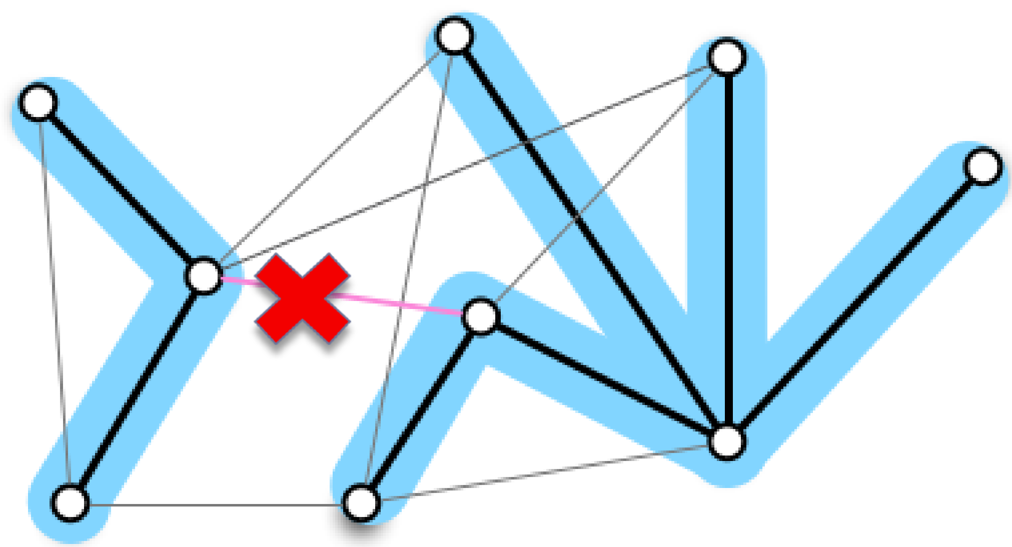

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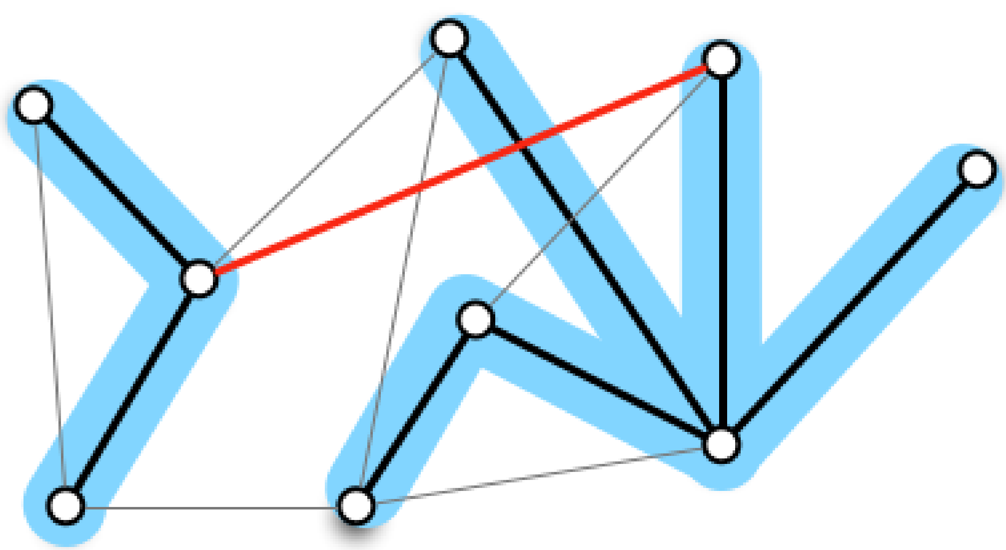
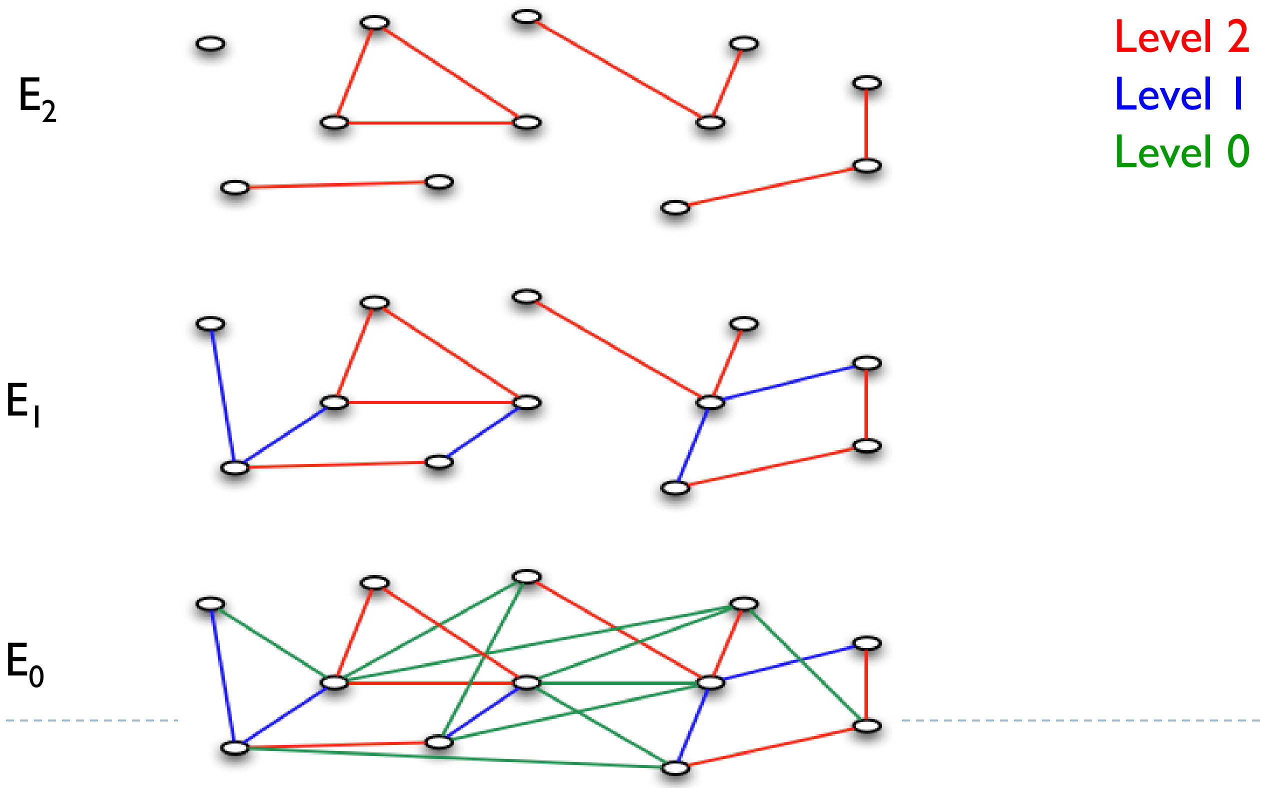


Holm, Lichtenberg & Thorup's structure

- $O(\log^2 n)$ amortized update time
- Best amortized update time so far.
- Appears in STOC'98

High-level description

- Each edge e is assigned a level l(e). ($0 \le l(e) \le l_{max}$)
- E_i={edges of level ≥ i}
- So $E=E_0 \supseteq E_1 \supseteq \ldots \supseteq E_{lmax}$



High-level description

- We keep the set of spanning forest $F=F_0 \supseteq F_1 \supseteq \ldots \supseteq F_{lmax}$ on $E_0, E_1, \ldots, E_{lmax}$
  - if $e=(u,v)$ is a non-tree edge in E_i, u and v are connected in F_i
  - if e is a tree edge in F_i, it must be a tree edge in F_j $(j<i)$
- Also, the number of vertices of a tree in F_i is at most $n/2^i$
- These properties are maintained throughout the algorithm.

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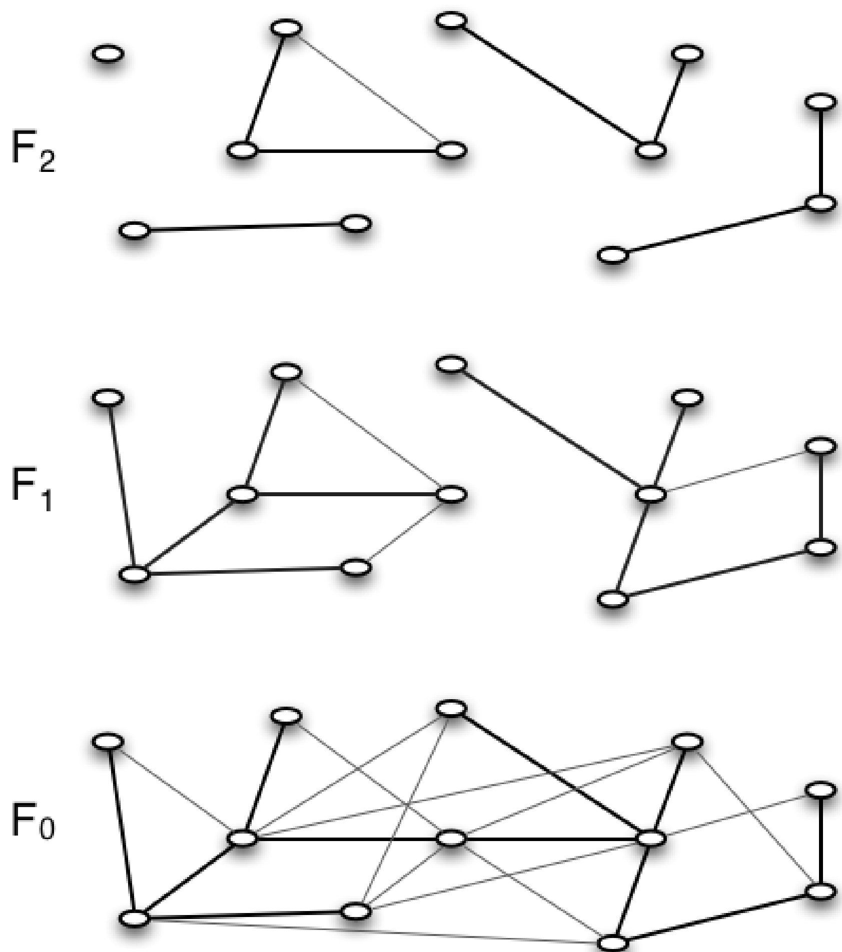
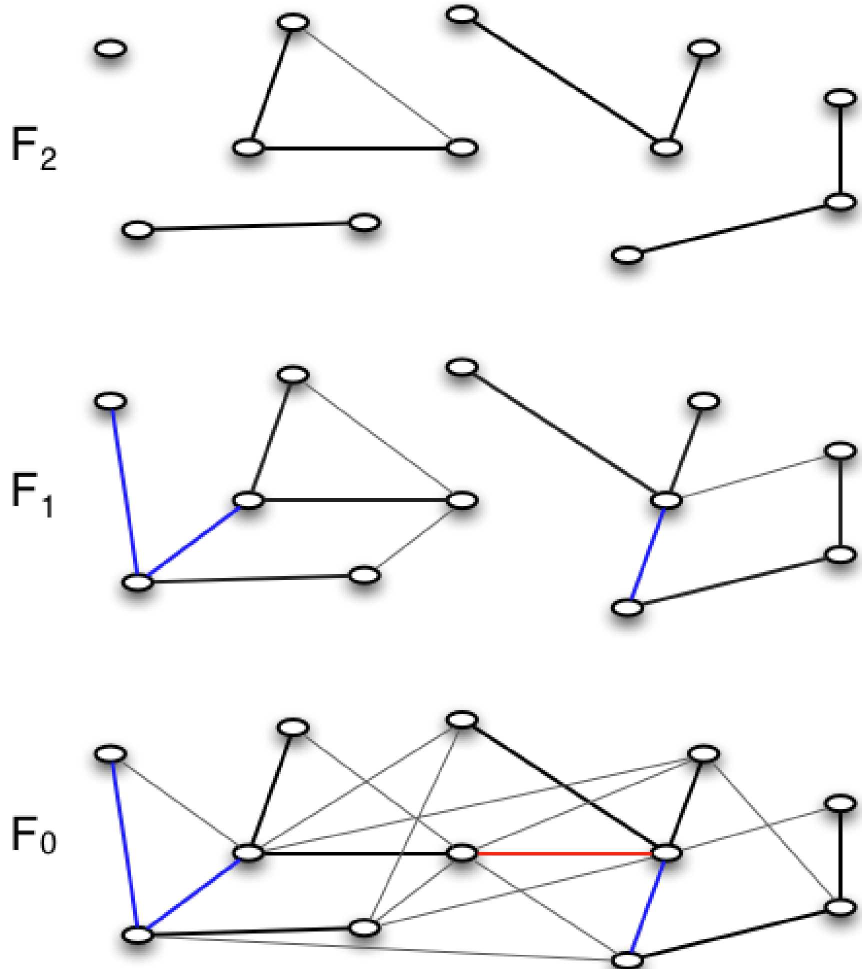
- Also, the number of vertices of a tree in F_i is at most $n/2^i$
  - The sizes of connected components decrease by a half when level increases
  - So $l_{max}=O(\log n)$
- These properties are maintained throughout the algorithm.

Example



Example – tree edge

- level ≥2
- level 1
- level 0



Remind

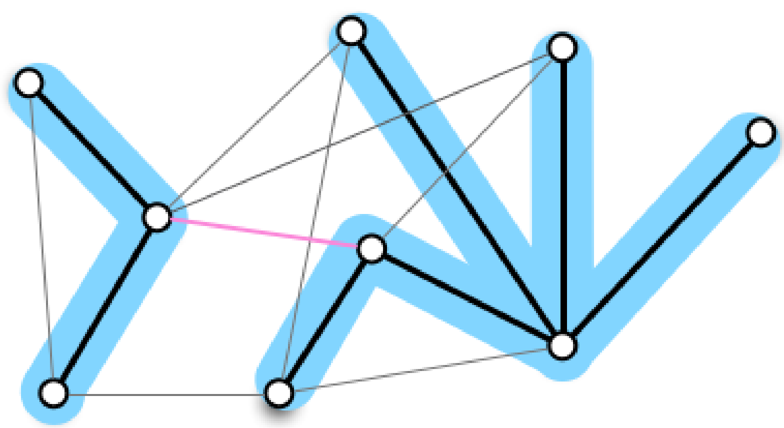
- Inserting (u,v):
  - When u,v are in the same tree, F do not change
  - When u,v are not in the same tree, connect these trees to a bigger tree
- Deleting a tree edge (u,v):
  - The tree will be split into two parts
  - We need to find other edges reconnecting these two parts

Algorithm

- Initially the graph is empty
- Level of an edge only increases, never decreases
  - When we have checked the edge, its level increases
  - Only increases for l_{max}=O(log n) times
  - So the amortized time for an edge is very small.

Algorithm

- Insert(e):
  - $l(e)=0$, if its two ends are not connected in F_0, e is added to F_0
- Delete(e):
  - If e is not a tree edge at level $l(e)$, simply delete e
  - If e is a tree edge, delete it in $F_0, F_1, \ldots, F_{l(e)}$, and call Reconnect(e, l(e))



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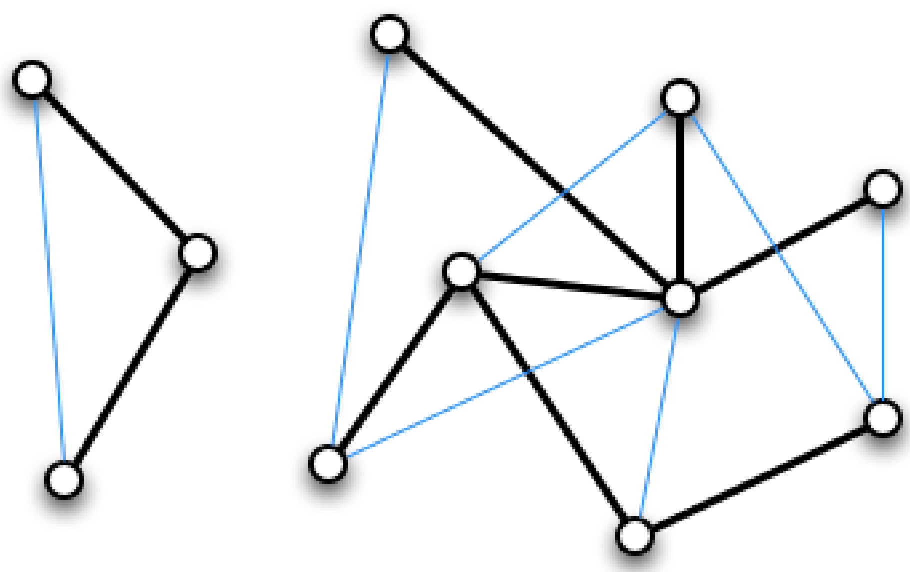
Spanning forests $F=F_0 \supseteq F_1 \supseteq \ldots \supseteq F_{lmax}$ on $E_0, E_1, \ldots, E_{lmax}$
So when e is not a tree edge at its level $l(e)$, it can not be a tree edge at other levels.

Algorithm

- Reconnect((u,v),i) – reconnect trees containing u and v by edges of level i
  - T– original tree in F_i containing (u,v),
  - T(u),T(v)– trees in F_i containing u,v after deletion of (u,v)
  - One of T(u),T(v) has at most a half as many vertices as T, assume it is T(u), move T(u) to level i+1
  - Check level i edges f incident to T(u) one by one, either:
    - f does not connect T(u) and T(v), then it must be included in T(u), increase its level to i+1
    - f connect T(u) and T(v), stop the search, and add f to $F_0, F_1, \ldots, F_i$
  - If no such edges are found, call Reconnect((u,v),i-1)
  - If i=0, we conclude that there is no reconnecting edges.

Algorithm

- f does not connect T(u) and T(v), then it must be included in T(u), increase its level to i+1 (since $|T(u)| \leq ½|T|$)
- f connect T(u) and T(v), stop the search, and add f to $F_0, F_1, \ldots, F_i$



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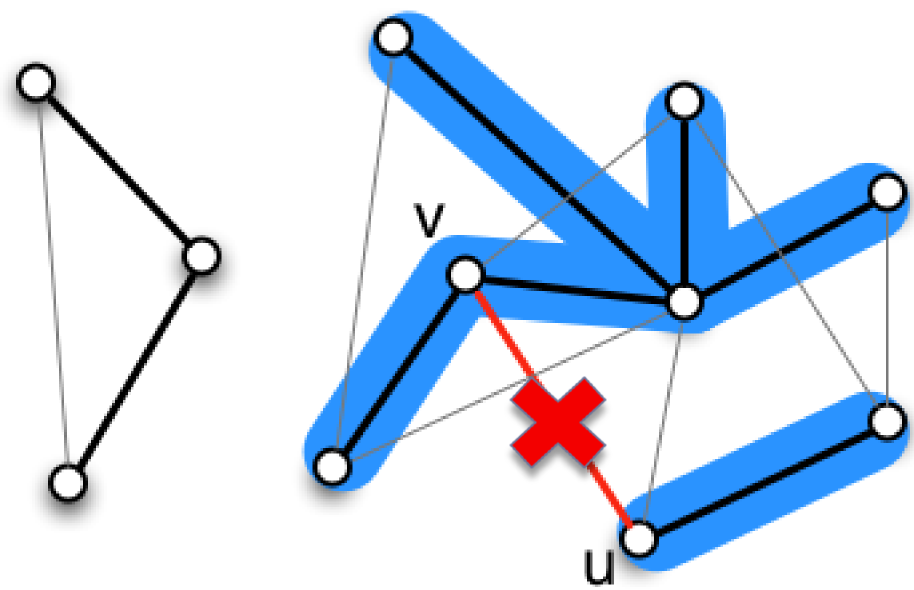

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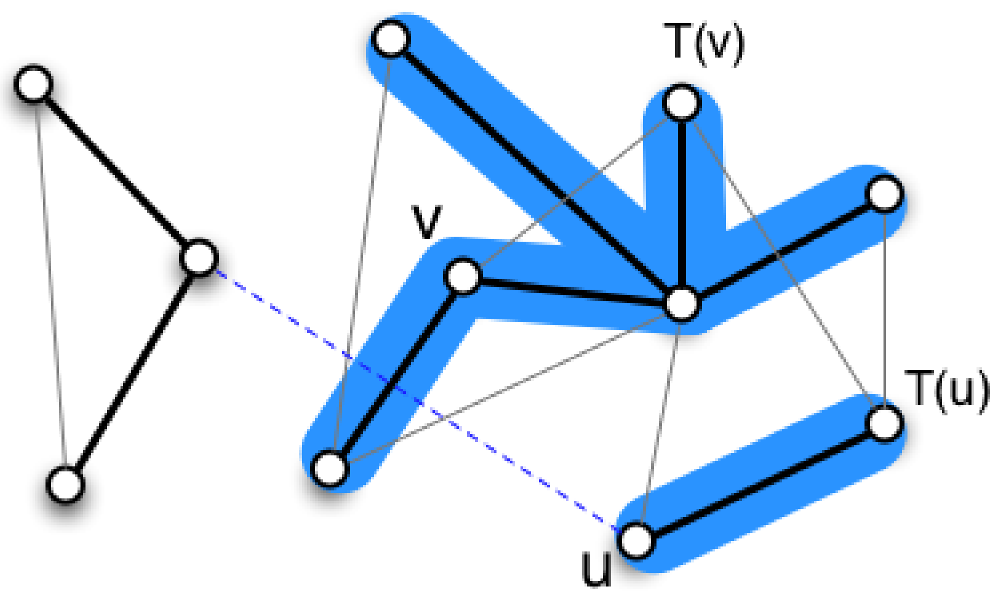


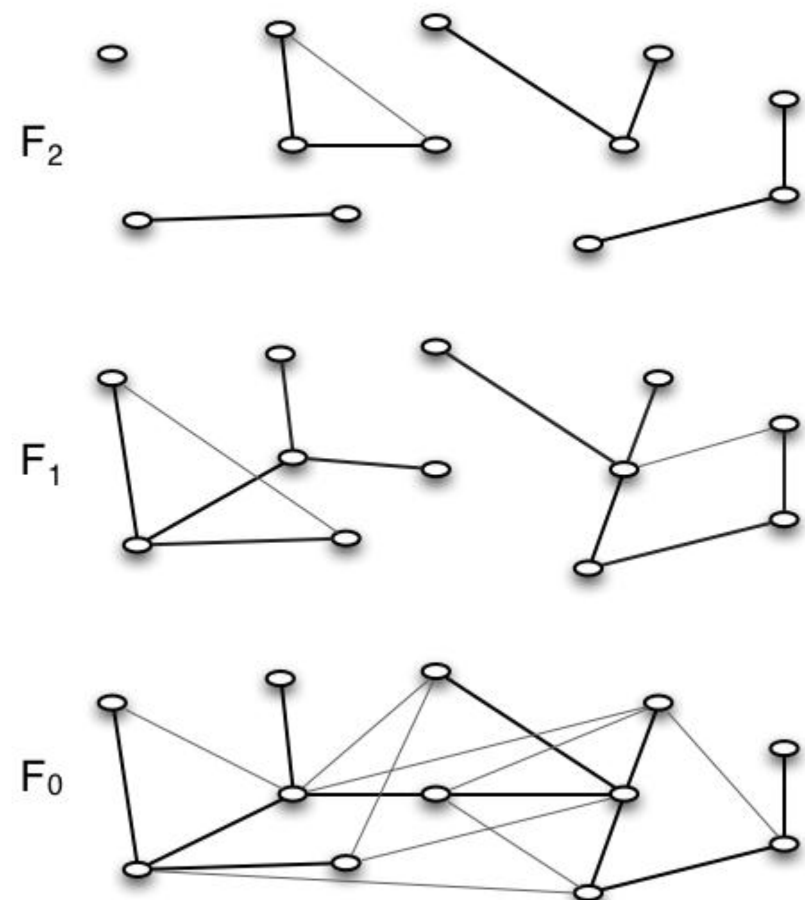
Bound the reconnecting time

- In one update we may need to check all the edges associated with a subtree T(u)
- But after checking an edge, its level increases, so every edge can be checked O(log n) times
- If initially the graph is empty, the number of edges is at most the number of update, so we need to check O(log n) edges per update.

- We keep the set of spanning forest $F=F_0 \supseteq F_1 \supseteq \ldots \supseteq F_{lmax}$ on $E_0,E_1,\ldots,E_{lmax}$
  - if $e=(u,v)$ is a non-tree edge in E_i, u and v are connected in F_i
  - if e is a tree edge in F_i, it must be tree edge in F_j $(j<i)$
- Also, the number of vertices of a tree in F_i is at most $n/2^i$
- These properties hold after the update algorithm

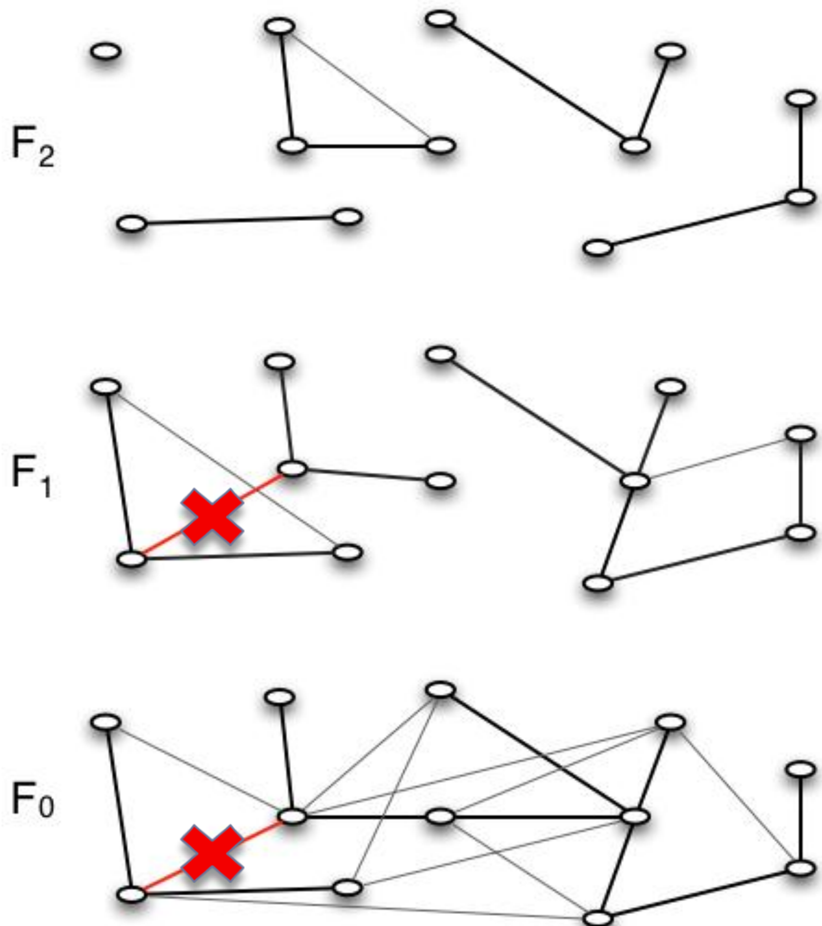
Example

- F_0, F_1, F_2: (non-tree edges are shown only in their levels)



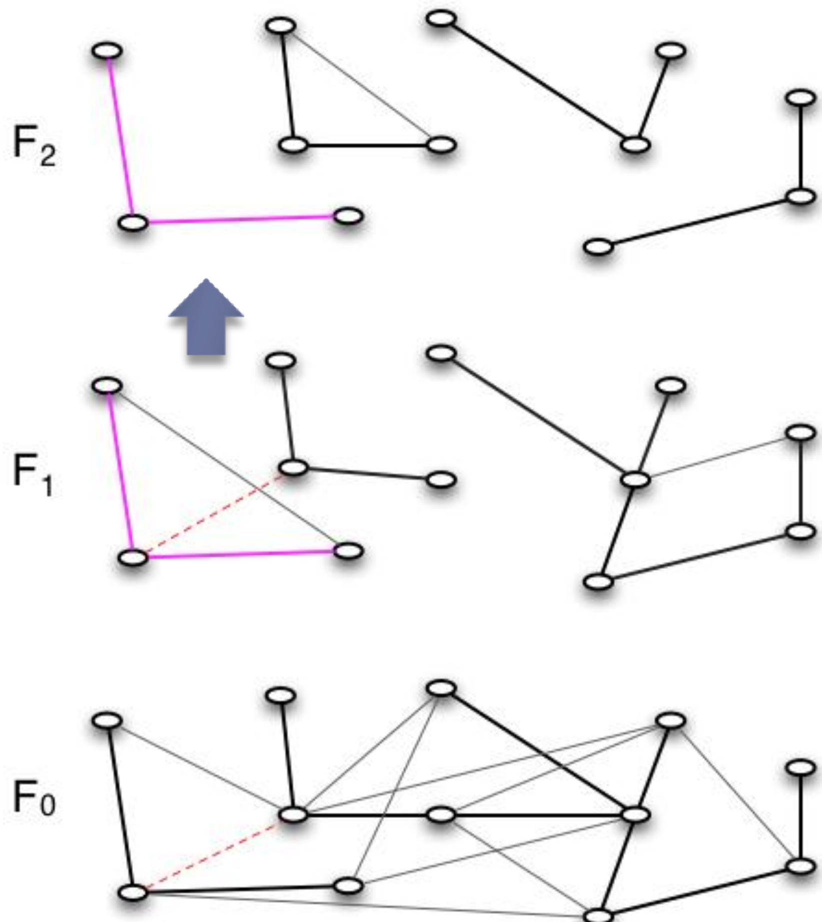
Example

- Deleting a tree edge:



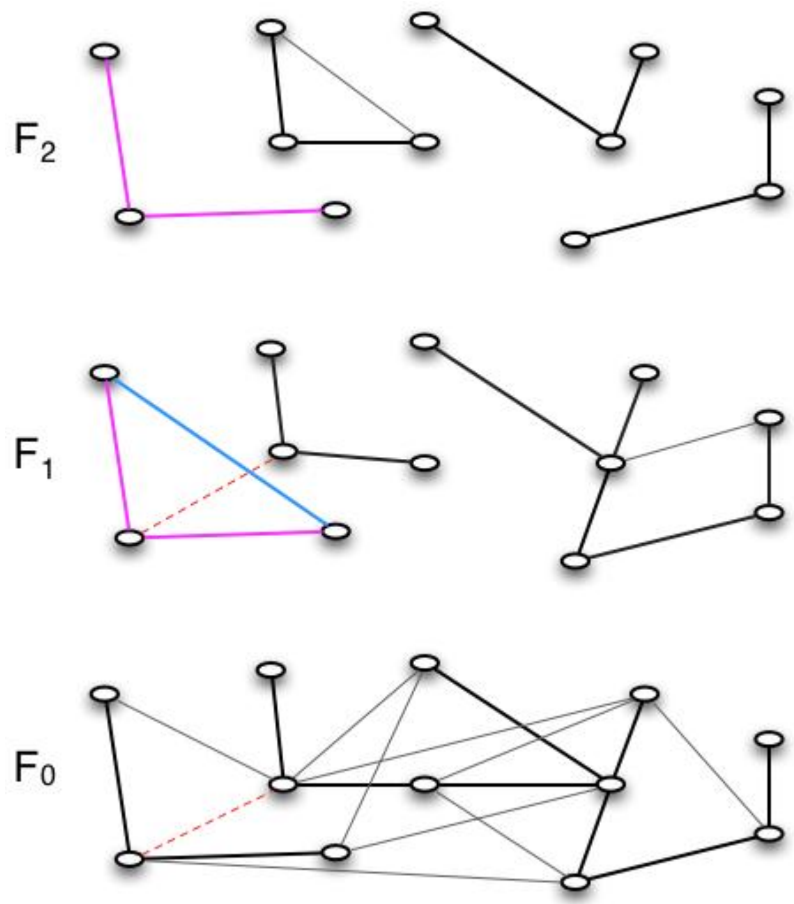
Example

- Call Reconnect(e,l(e))



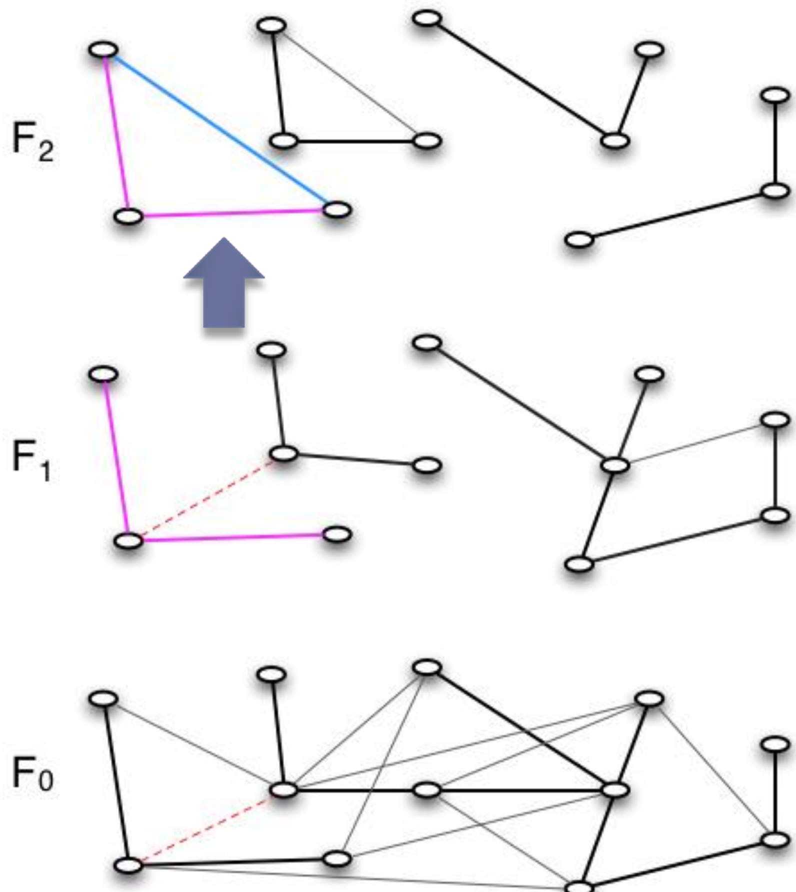
Example

- Check for an edge whether it can reconnect them



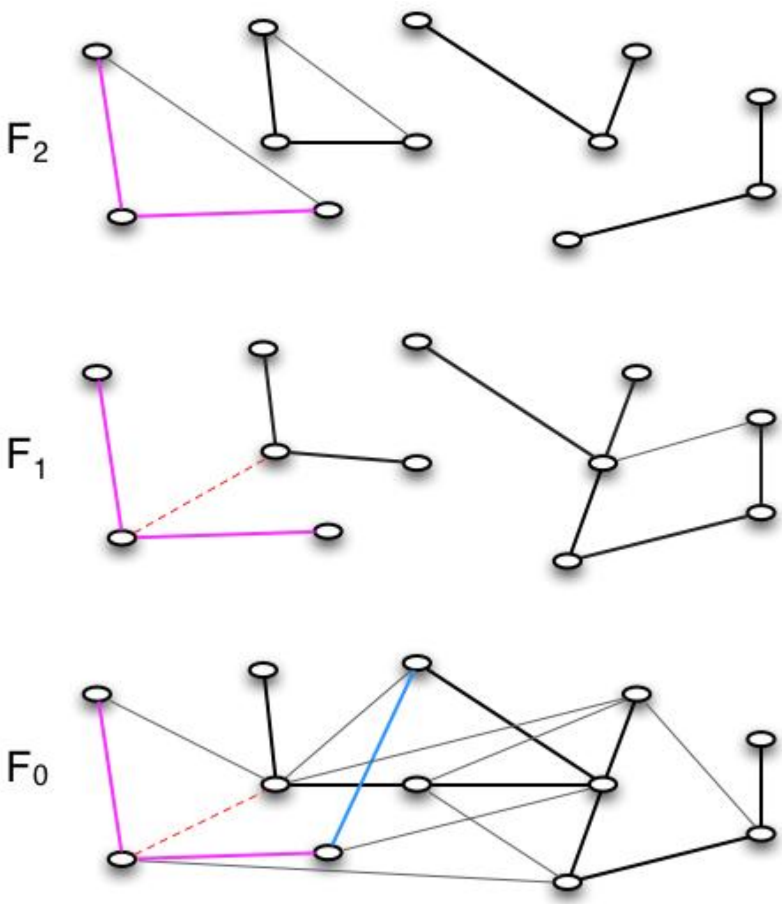
Example

- Remove it to higher level



Example

- Call reconnect in lower level



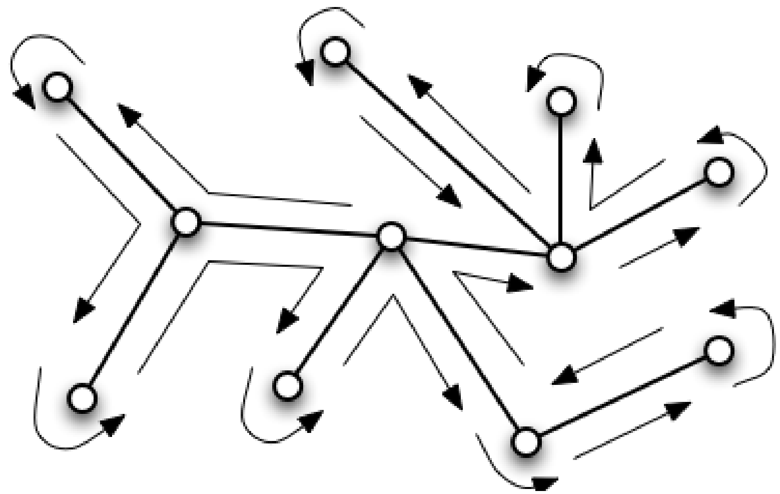
Implementation

- We need to keep dynamic forest
  - Merge two tree by an edge
  - Split a tree into two subtrees
  - Find the tree containing a given vertex
  - Return the size of a tree
  - Min-key: returns the minimal key in a tree

- These operations can all be done in O(log n) time.

ET-trees

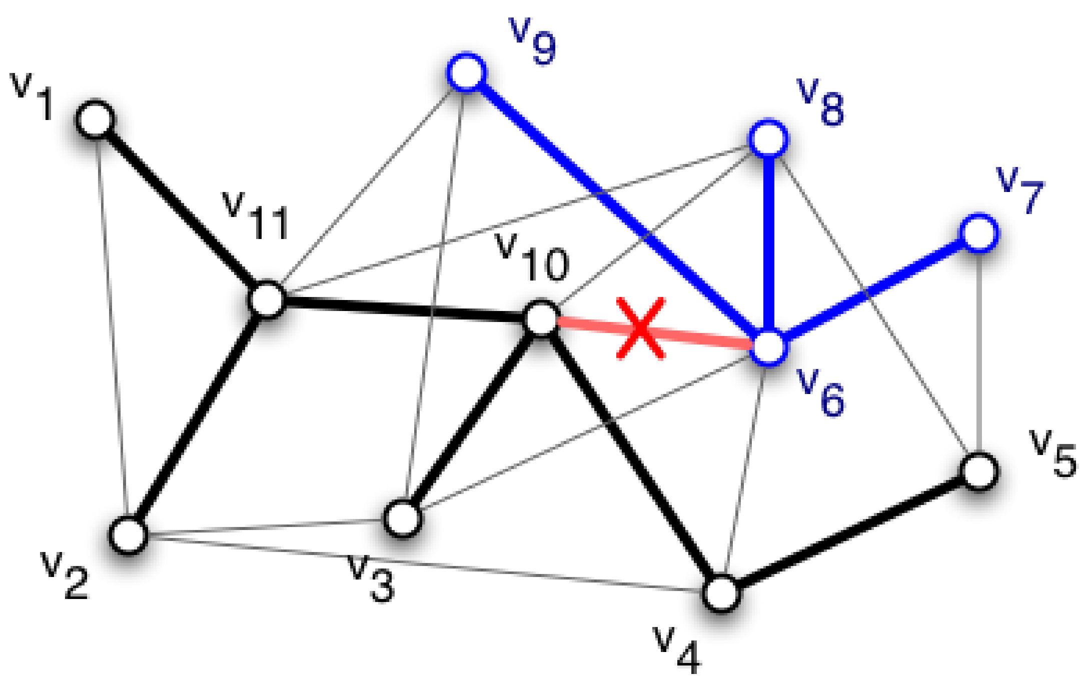
- Euler Tour of T:



- Every vertex can appear many times in the Euler Tour, but we only keep any one of them for each vertex to form a ET-list:

  $v_1, \ v_2, \ \ldots v_n$

When we delete a tree edge, the ET-list will be divided into ≤3 parts, and we need to merge two lists.

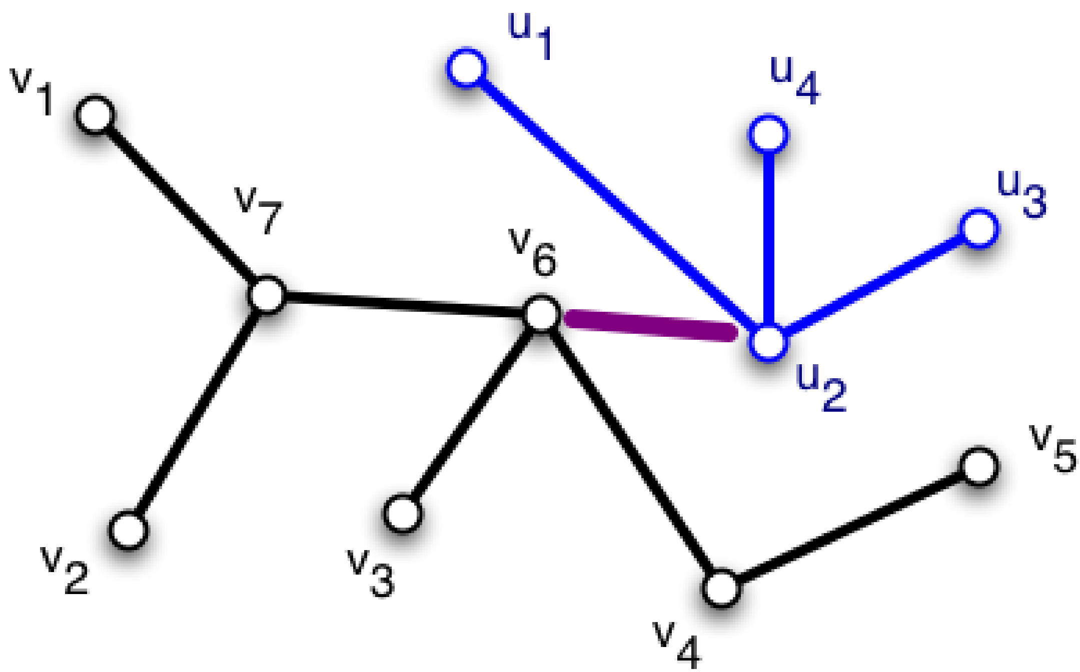


$v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}$

↓

$(v_1, v_2, v_3, v_4, v_5, v_{10}, v_{11})$; (v_6, v_7, v_8, v_9)

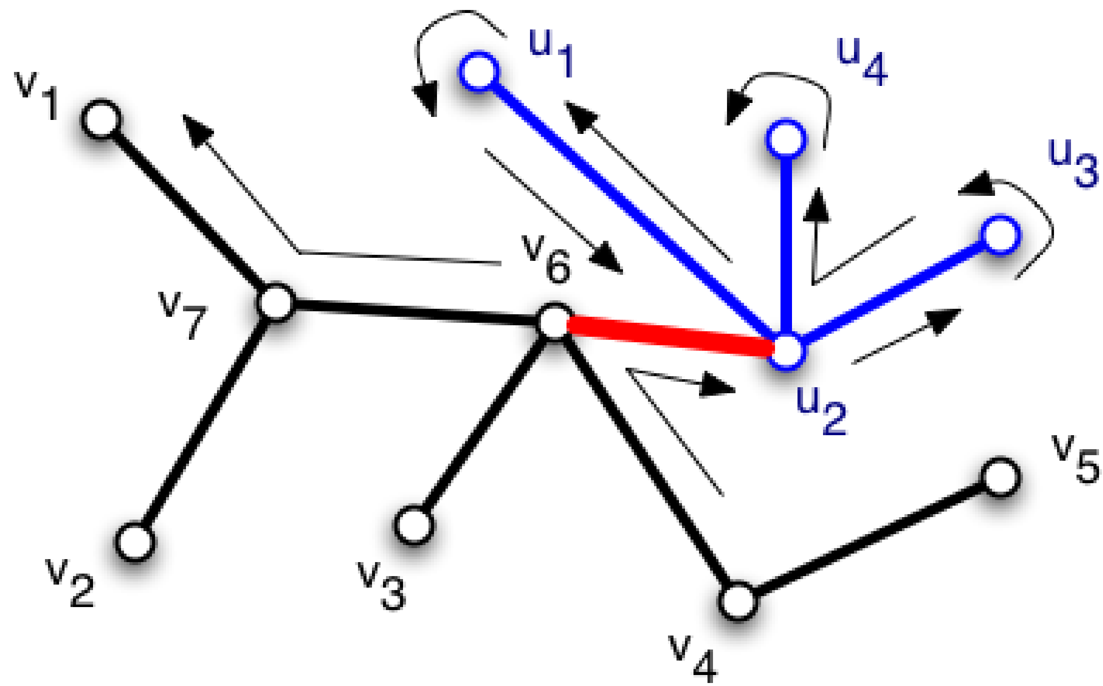
When we connect two trees by an edge, we need to split the ET-lists of the two trees from the vertices on that edge …



$(v_1, \; v_2, \; v_3, \; v_4, \; v_5), \; (v_6, \; v_7)$

$(u_1), \; (u_2, \; u_3, \; u_4)$

When we connect two trees by an edge, we need to split the ET-lists of the two trees from the vertices on that edge, and merge them in the right order.



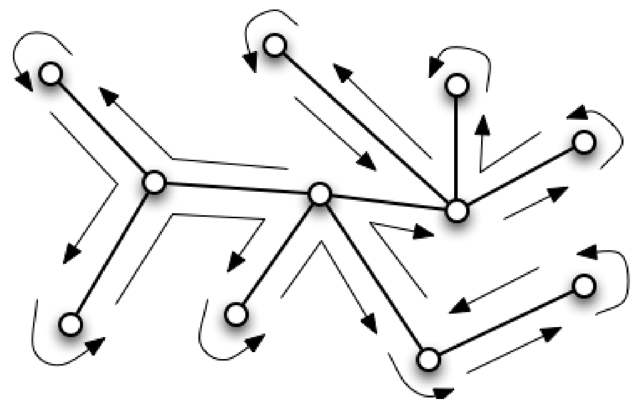
$(v_1, v_2, v_3, v_4, v_5), (v_6, v_7); (u_1), (u_2, u_3, u_4)$

$\Downarrow$

$(v_1, v_2, v_3, v_4, v_5, u_2, u_3, u_4, u_1, v_6, v_7)$

Euler Tour

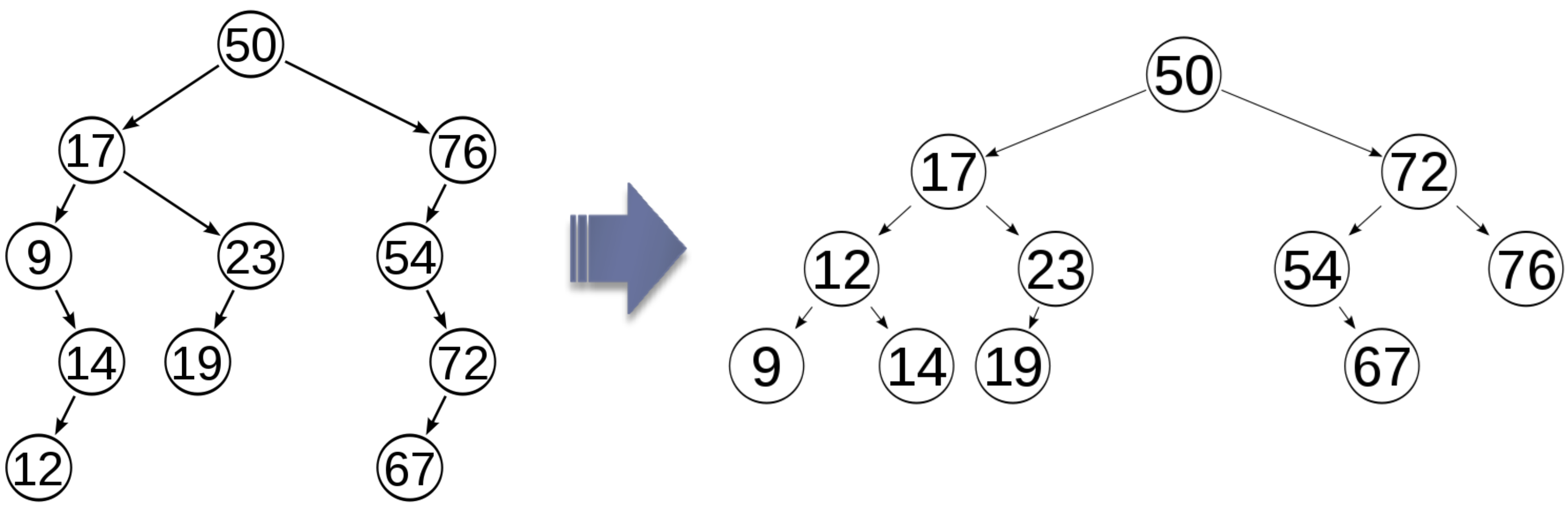
- Euler Tour of T:



- So we only need O(1) link & cut operations to maintain the ET-lists per tree merging or splitting.
- However, we need balanced binary trees to keep the ET-lists, so it takes O(log n) time to rebalancing after a update,

Self-balancing binary search tree

- Automatically keep its height O(log n)



Self-balancing binary search tree

- Need O(log n) time to rebalancing
- O(log n) time to find the root from a vertex
- Every vertex can store the size or min-key of its subtree, so these information can be maintained in O(log n) time per update.

ET-tree

- We need to keep dynamic forest
  - Merge two tree by an edge
  - Split a tree into two subtrees
  - Find the tree containing a given vertex
  - Return the size of a tree
  - Min-key: returns the minimal key in a tree
- These operations can all be done in O(log n) time.

Back to dynamic connectivity

- If initially the graph is empty, the number of edges is at most the number of update, so we need to check O(log n) edges per update.
- Since merging two trees takes O(log n) time, and an edge can merge trees in O(log n) levels, so the amortized update time is $O(\log^2 n)$

Back to dynamic connectivity

- If initially the graph is empty, the number of edges is at most the number of update, so we need to check O(log n) edges per update.
- Since merging two trees takes O(log n) time, and an edge can merge trees in O(log n) levels, so the amortized update time is $O(\log^2 n)$.
- Deletion can cost $O(\log^2 n)$ time.
  - Delete an edge in l_{max} trees

- Query time: O(log n/loglog n)
- Space: O(m+nlog n) (almost linear)

Dynamic Minimum Spanning Tree

- Much more complicated since we need to consider the order of edges
- Decremental minimum spanning tree
  - Only a modification from dynamic connectivity structure
  - Only support deletions

Algorithm

- Originally we have a MST F_0 at level 0
- Delete(e):
  - If e is not a tree edge at level l(e), simply delete e
  - If e is a tree edge, delete it in $F_0, F_1, \ldots, F_{l(e)}$, and call Reconnect(e, l(e))

Spanning forests $F=F_0 \supseteq F_1 \supseteq \ldots \supseteq F_{lmax}$ on $E_0, E_1, \ldots, E_{lmax}$
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Algorithm

- Reconnect((u,v),i) – reconnect trees containing u and v by edges of level i
  - T– original tree containing (u,v),
  - T(u),T(v)– trees containing u,v after deletion of (u,v)
  - One of T(u),T(v) has at most a half as many vertices as T, assume it is T(u), move T(u) to level i+1
  - Check level i edges f incident to T(u) **in increasing order**,
    - f does not connect T(u) and T(v), then it must be included in T(u), increase its level to i+1
    - f connect T(u) and T(v), stop the search, and add f to F_0, F_1,…,F_i
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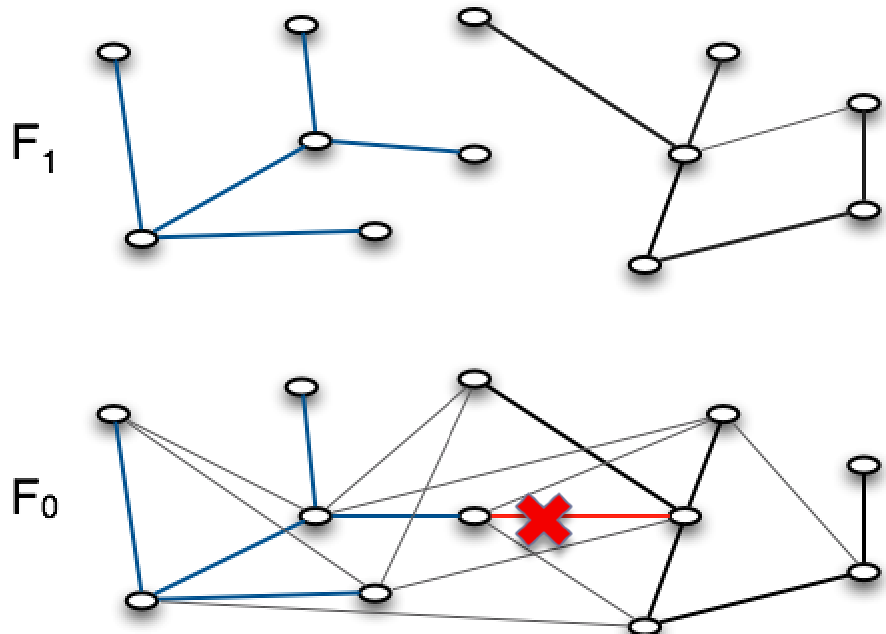
  - Intuitively, we can see we find the minimum edge which reconnects the two subtrees.

Example



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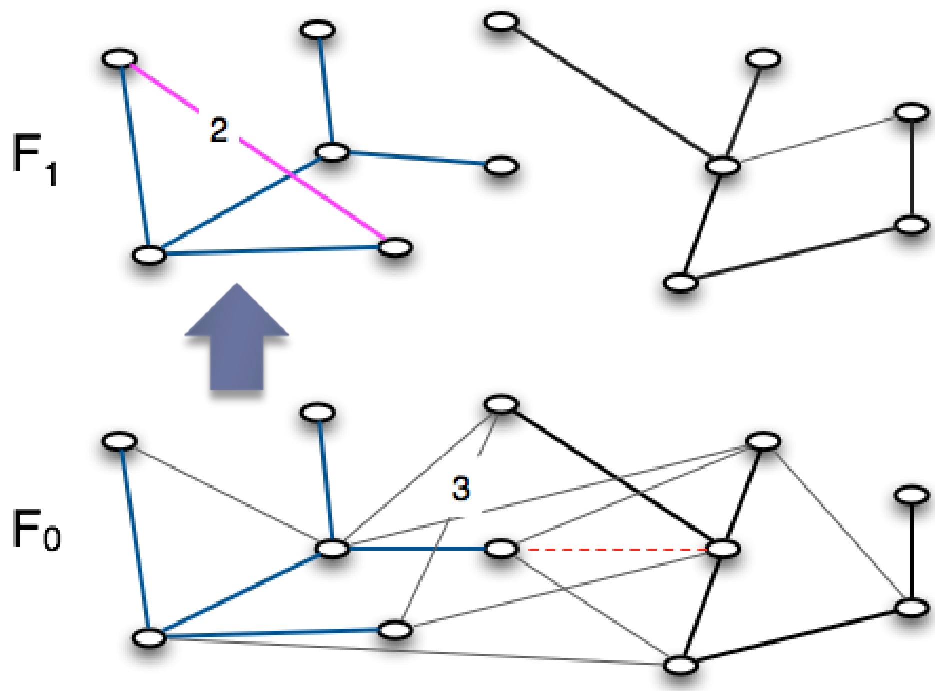

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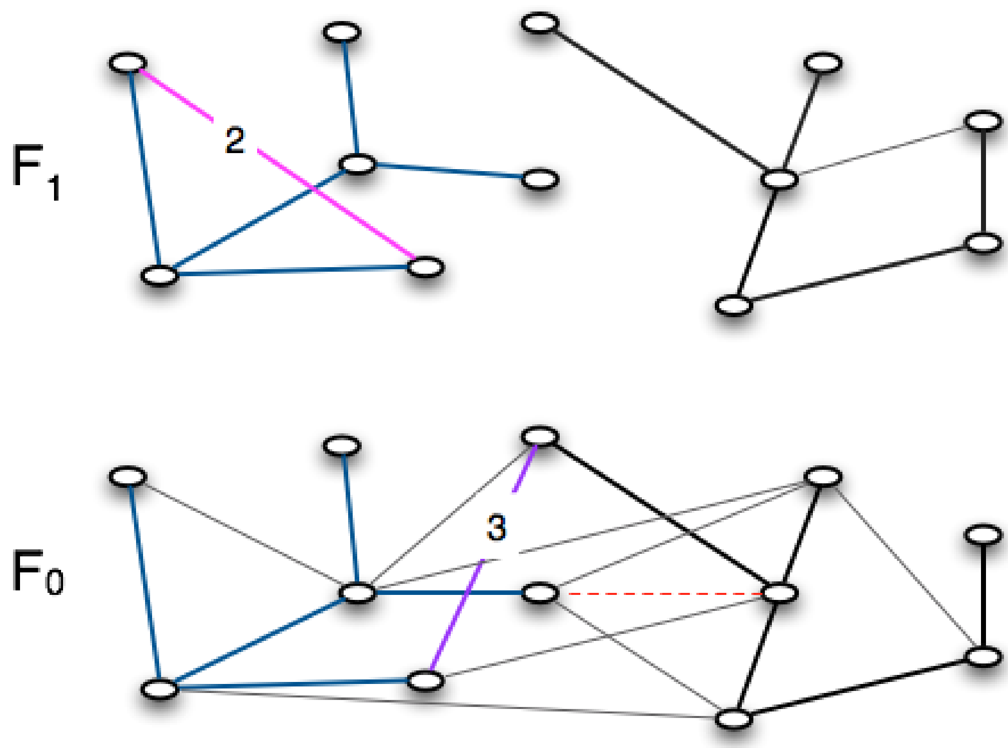
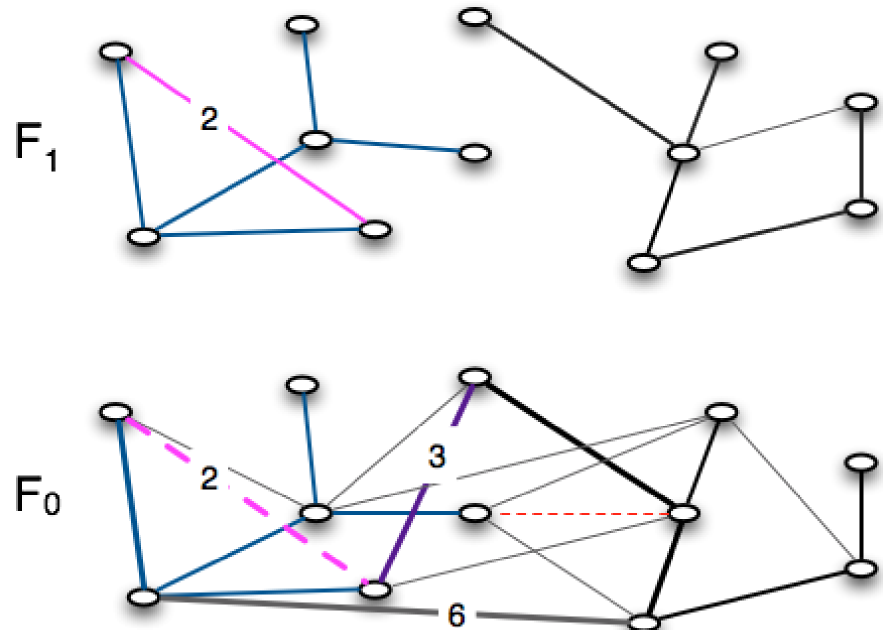


Invariants

1. We keep the set of spanning forest $F=F_0 \supseteq F_1 \supseteq \ldots \supseteq F_{lmax}$ on $E_0, E_1, \ldots, E_{lmax}$
2. The number of vertices of a tree in F_i is at most $n/2^i$
3. Every cycle C has a non-tree edge e with:

$$w(e) = \max_{f \in C} w(f)$$

$$l(e) = \min_{f \in C} l(f)$$



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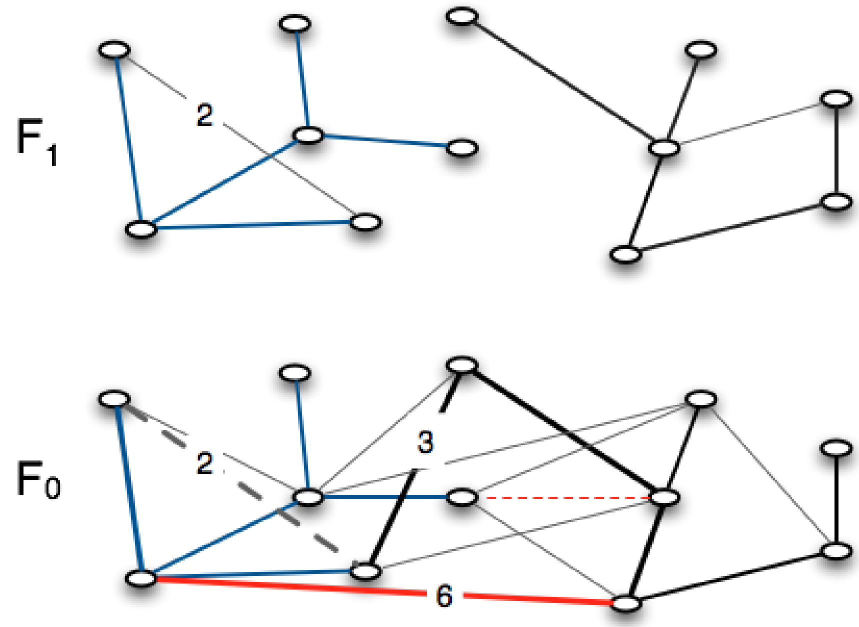


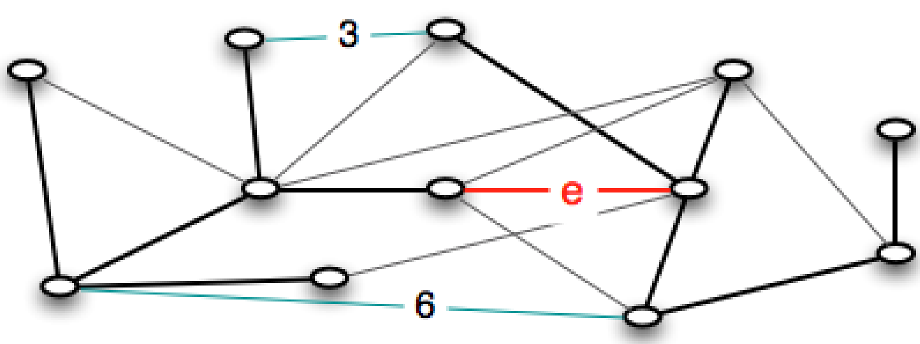
Proof of correctness

- Assume (3), the lightest replacement edge is on the maximum level

- The algorithm maintains (3)

Proof of correctness

- Assume (3), the lightest replacement edge is on the maximum level
  - Compare two replacement edges e_1, e_2, if $w(e_1) \le w(e_2)$, we need to prove $l(e_1) \ge l(e_2)$
  - e_1, e_2 can form cycles C_1, C_2 with the original tree



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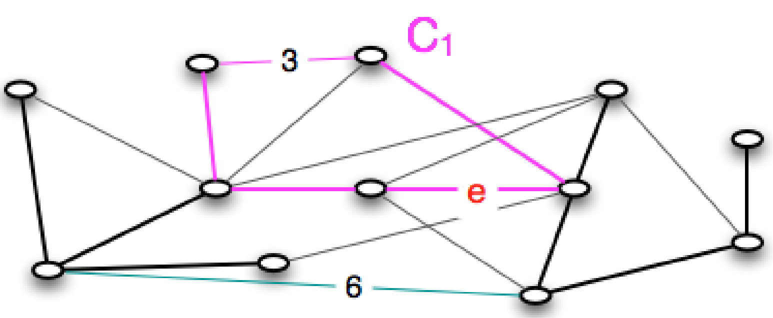
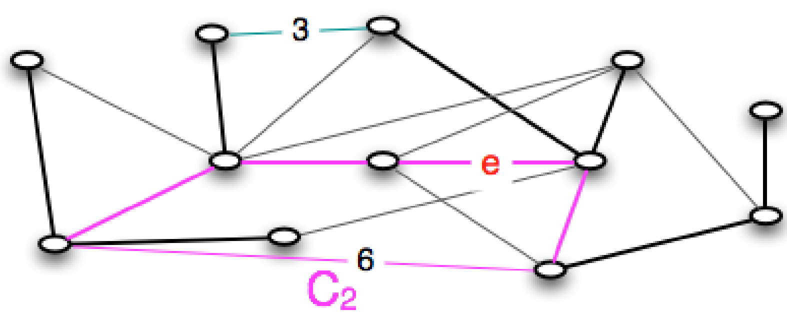
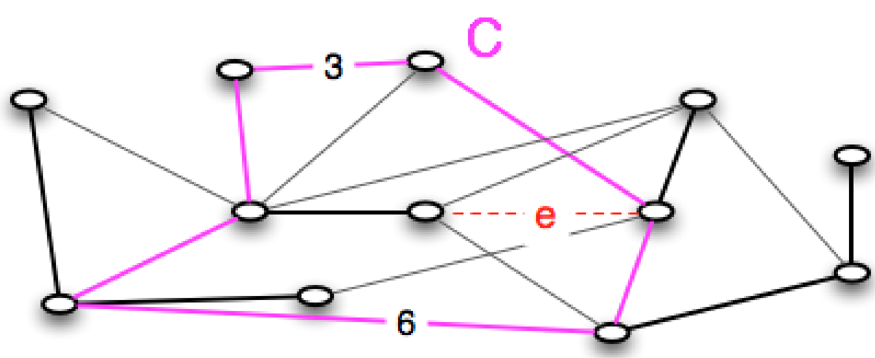
  - Compare two replacement edges e_1, e_2, if $w(e_1) \le w(e_2)$, we need to prove $l(e_1) \ge l(e_2)$
  - e_1, e_2 can form cycles C_1, C_2 with the original tree
    - e_1, e_2 must be largest edges in C_1, C_2, resp. Otherwise original tree is not minimum





- Assume (3), the lightest replacement edge is on the maximum level
  - Compare two replacement edges e_1, e_2, if $w(e_1)<w(e_2)$, we need to prove $l(e_1) \geq l(e_2)$
  - e_1, e_2 can form cycles C_1, C_2 with the original tree
    - e_1, e_2 must be largest edges in C_1, C_2, resp. Otherwise original tree is not minimum
  - $C=C_1 \oplus C_2$ is also a cycle with e_1 and e_2, and $w(e_2)$ is the largest in C, so $l(e_2)$ is lowest.



(3) Every cycle C has a non-tree edge e with largest weight and lowest level

Proof of correctness

- The algorithm maintains (3):

- When the level of e increases, e is in T(u)
  - Assume e is the unique lowest largest edge on some cycle C

  - All other edges of C incident to T(u) have level >l(e)
  - C cannot leave T(u)
  - So all other edges in C have level >l(e), so (3) is maintained when l(e) increases by 1

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  - All other edge of C incident to T(u) have level >l(e)
  - C cannot leave T(u) (Otherwise there will be a replacement found.)
  - So all other edges in C have level >l(e), so (3) is maintained when l(e) increases by 1

Update time

- Only need to maintain min-key in ET-tree structure

- Update time for this decremental MST is still $O(\log^2 n)$

Discussion

- Why is it hard to extend this to fully dynamic MST?
  - Unlike connectivity structures, we may need to change the forest when inserting an edge.
  - Totally breaking the order of the structure

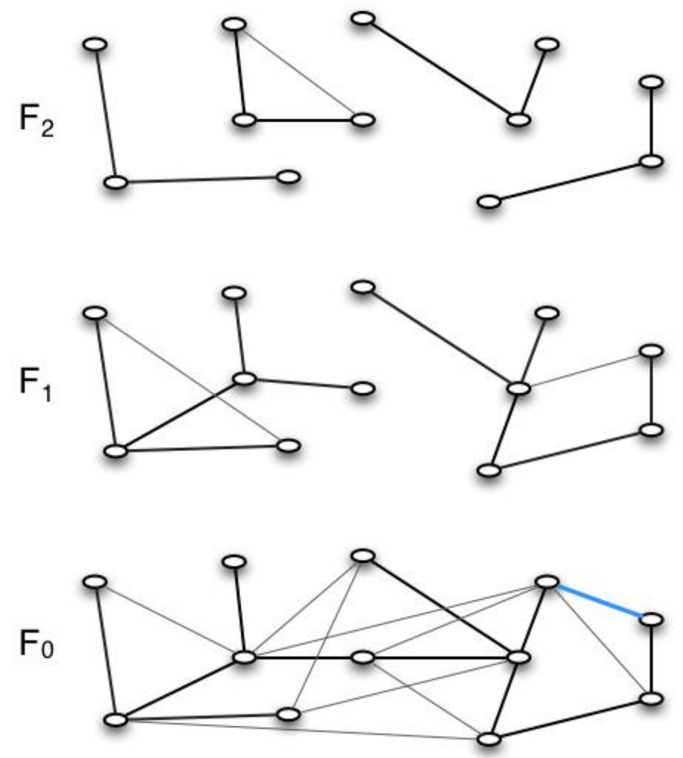
Invariants

1. We keep the set of spanning forest $F=F_0 \supseteq F_1 \supseteq \ldots \supseteq F_{lmax}$ on $E_0, E_1, \ldots, E_{lmax}$
2. The number of vertices of a tree in F_i is at most $n/2^i$
3. Every cycle C has a non-tree edge e with:

$$w(e) = \max_{f \in C} w(f)$$

$$l(e) = \max_{f \in C} l(f)$$

- *If we insert an edge with very small weight:*



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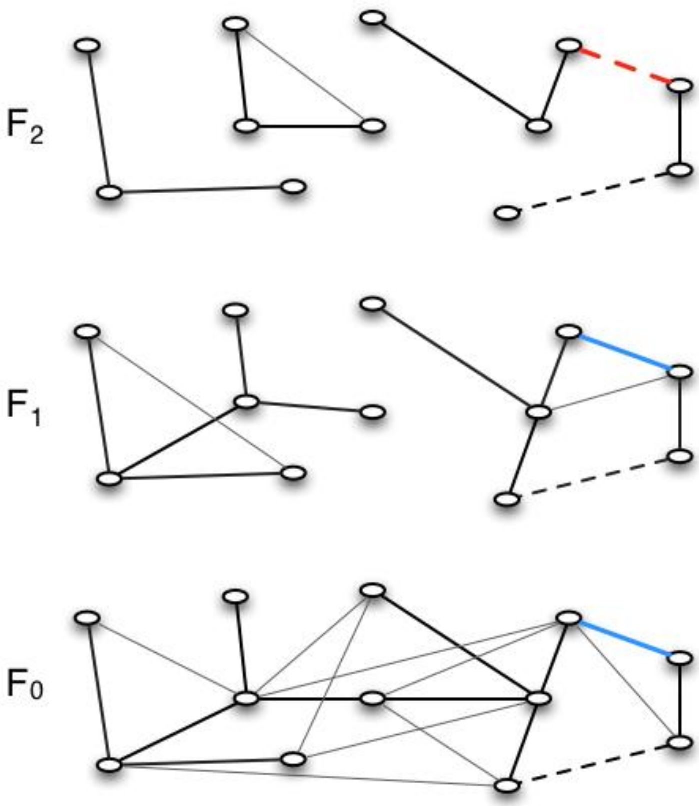


- *If we insert an edge with very small weight:*
- *The MST will change, so as MST in higher levels*

Invariants

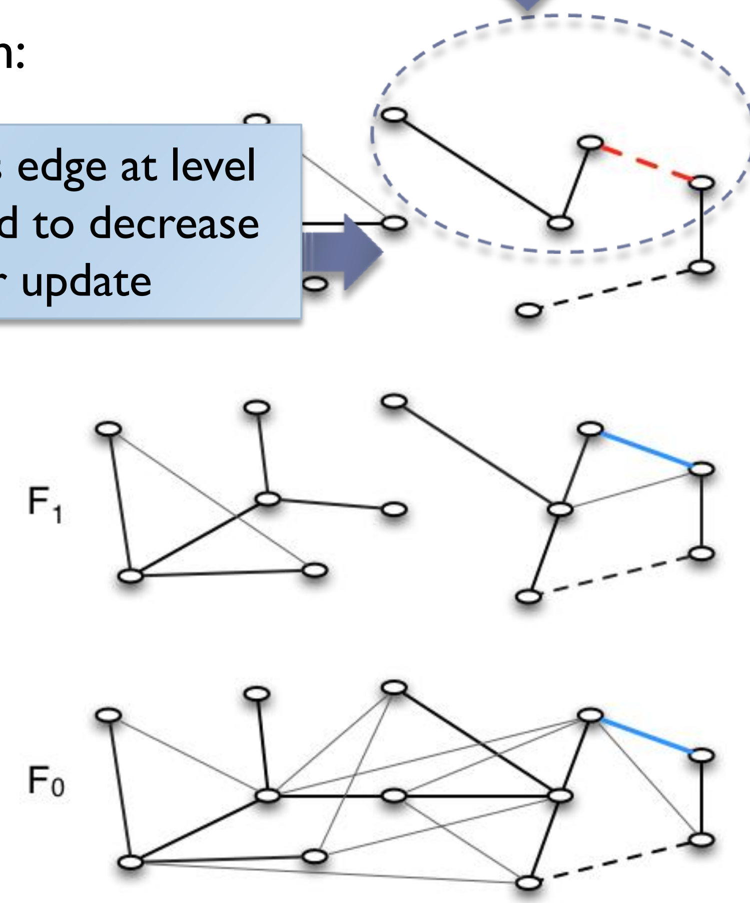
Too large for this level if we add the new edge here.

1. We keep the set of spanning forest $F=F_0 \supseteq F_1 \supseteq \ldots \supseteq F_{lmax}$
2. The number of vertices of a tree in F_i is at most $n/2^i$
3. Every cycle C has a non-tree edge e with:

$$w(e) = \max_{f \in C} w(f)$$

$$l(e) = \max_{f \in C} l(f)$$

Originally this edge at level 2, but we need to decrease this level after update



- *If we insert an edge with very small weight:*
- *The MST will change, so as MST in higher levels*
- *Level decreasing will destroy the hierarchy*

Fully dynamic MST

- An $O(\log^4 n)$ amortized update time structure is given in:
  - *"Poly-logarithmic deterministic fully-dynamic algorithms for connectivity, minimum spanning tree, 2-edge, and biconnectivity"*
  - By Holm, Lichtenberg, Thorup, **Jounral of ACM 2001**

  - Construct smaller decremental structure every time
  - Complicated analysis

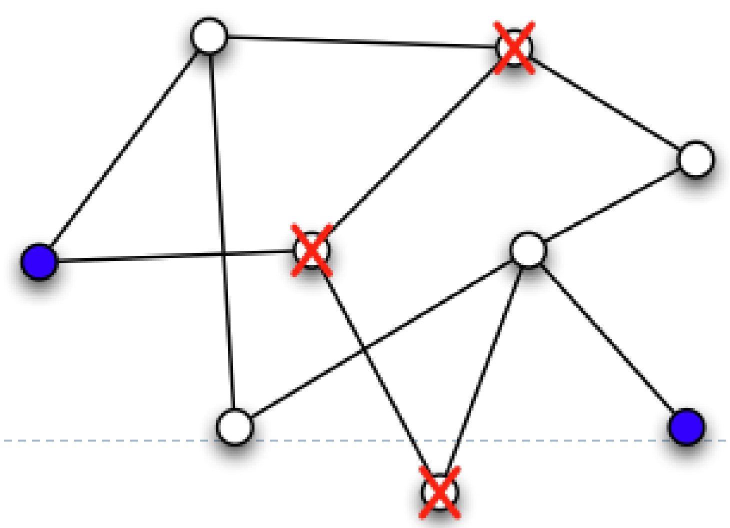
Overview of Dynamic Connectivity Results

- Edge update—amortized time
  - Holm, Lichtenberg, and Thorup: $O(\log^2 n)$

- Edge update—worst-case
  - Frederickson, Eppstein et al: $O(n^{1/2})$

Dynamic Subgraph Model

- There is a fixed underlying graph G, every vertex in G is in one of the two states "on" and "off".
- Construct a dynamic data structure:
  - Update: Switch a vertex "on" or "off".
  - Query: For a pair (u,v), answer connectivity/shortest path between u and v in the subgraph of G induced by the "on" vertices.

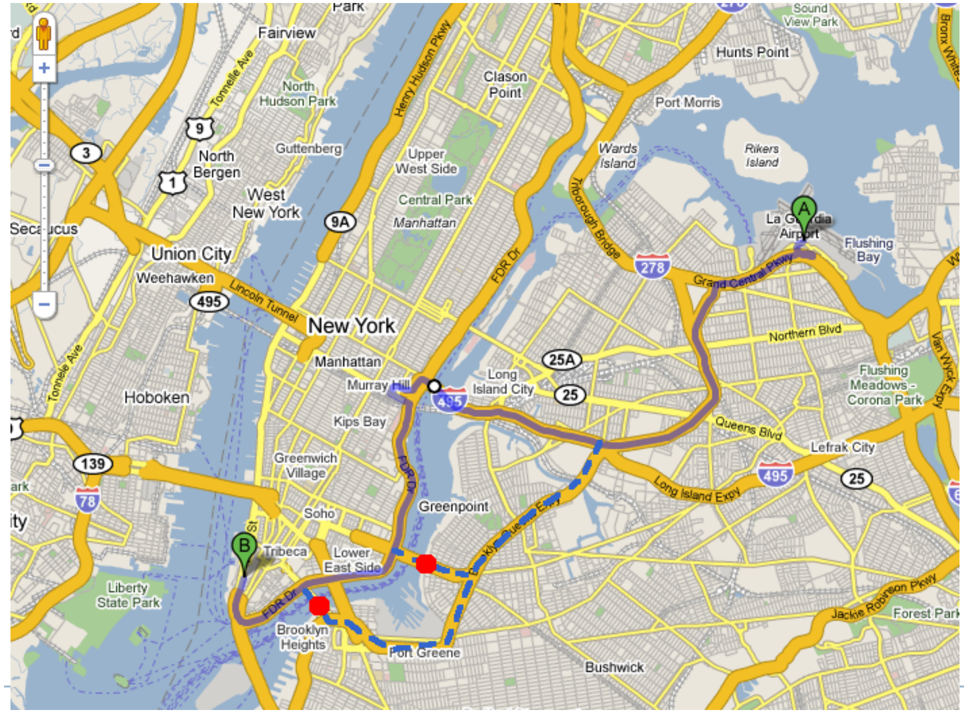


Dynamic Connectivity

	Edge Updates	Vertex Updates (Subgraph)
Amortized	$O(\log^2 n)$ [Holm, Lichtenberg & Thorup '1998]	$\tilde{O}(m^{2/3})$, with query time $\tilde{O}(m^{1/3})$ [Chan, Pâtraşcu & Roditty '2008]
Worst-Case	$O(n^{1/2})$ [$O(m^{1/2})$ by Frederickson '1985] [Improved by Eppstein, Galil, Italiano, Nissenzweig '1992]	$\tilde{O}(m^{4/5})$, with query time $\tilde{O}(m^{1/5})$ [Duan 2010]

d-failure Model

- **d-failure model:**
  - The number of "failed" vertices/edges is bounded by d
  - It can be seen as a static structure, in which the query (u,v) is given with a set D of "failed" vertices/edges and |D|≤ d



Next lecture

- Worst-case dynamic connectivity