Matroid uses in kernelization

Magnus Wahlström
Joint work with Stefan Kratsch

Max Planck Institute for Informatics

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Matroids

A matroid $M = (V, \mathcal{I})$, $\mathcal{I} \subseteq 2^V$, is an independence system with independent sets $\mathcal{I}$ satisfying:

1. The empty set is independent
2. A subset of an independent set is independent
3. Augmentation property: If A, B are independent and $|B| > |A|$, then there is some $b \in B - A$ such that $A + b$ is independent

Rank $r(X)$: Size of largest independent subset of X

Examples

Canonical examples:

1. Graphic matroids: Let $G = (V, E)$ be a graph.
   - $M = (E, \mathcal{I})$, $\mathcal{I}$ contains cycle-free edge sets
   - Rank: number of vertices minus number of components
2. Linear matroids $M = (V, \mathcal{I})$:
   - V is a collection of vectors in $\mathbb{F}^d$ for some field $\mathbb{F}$
   - Independence concept is linear independence
   - Rank: dimension

Representable matroid: Can be given as set of vectors.

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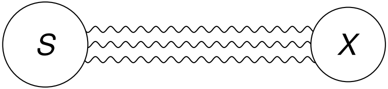
Graph cut matroids: Gammoids

Let $G = (V, E)$ be a graph (possibly directed). Say that $T \subseteq V$ is linked to S if there are $|T|$ vertex-disjoint paths from S to T (with distinct endpoints).

The gammoid (G, S) defined by G and S is $M = (V, \mathcal{I})$ where:

- $\mathcal{I}$ contains all sets linked to S
- The rank $r(X)$ equals the size of an (S, X)-cut



Gammoids are representable.

Matroids in kernelization

Matroid theory was integral to the following kernelization results:

- Odd Cycle Transversal polynomial compression (SODA 2012)
- Almost 2-SAT polynomial kernel
- Multiway Cut with Deletable Terminals
- Restrictions of cut problems: Multiway Cut with s terminals, Multicut with s requests, Group Feedback Vertex Set with fixed group

Covering terminal min-cuts

Further result: Cut-covering sets

Covering (A, B)-cuts

Let $G = (V, E)$, $S, T \subseteq V$, $S \cap T = \emptyset$. We can find a set $X \subseteq V$ of polynomial size such that for any $A \subseteq S$ and $B \subseteq T$, X contains a minimum (A, B)-cut.

Three uses of matroids

Matroids have seen three types of use in kernelization:

1. Encoding information succinctly
2. Sunflower-type constraints reductions
3. Irrelevant vertex rules

Three uses of matroids

Matroids have seen three types of use in kernelization:

1. Encoding information succinctly
   - ODD CYCLE TRANSVERSAL
2. Sunflower-type constraints reductions
   - ALMOST 2-SAT
3. Irrelevant vertex rules
   - *s*-MULTIWAY CUT, cut-covering sets applications

Matroids for encoding

- Matroid $M = (V, \mathcal{I})$, represented by matrix A: $r(M) \cdot |V| \cdot \ell$ space, bitlength ℓ
- One known use: Gammoids encoding terminal cut functions
  - Input: Graph $G = (V, E)$, terminals $T \subset V$, need to know size of all min-cuts through T
  - $\mathcal{O}(|T|) \times \mathcal{O}(|T|)$ matrix, $\ell = \mathcal{O}(|T|)$ (Kratsch, W., SODA 2012)
  - Now known: Also encodable as graph with $\mathcal{O}(|T|^3)$ vertices.

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Representative sets

For advanced applications, we need the following notions.

Let $M = (V, \mathcal{I})$ be a matroid, $X \subseteq V$.

- Let $t = \{t_1, \ldots, t_s\} \subset V$. We say that t extends X if $X \cap t = \emptyset$ and $X \cup t$ is independent (i.e., $r(X \cup t) = |X| + |t|$).
- Let $T \subseteq \binom{V}{s}$. We say that T extends X if t extends X, $t \in T$.
- A set $T^* \subseteq T$ represents T in M if, for any X, T extends X if and only if T^* extends X.

Representative sets (Marx (2006) using Lovász (1977))

Let $M = (V, \mathcal{I})$ be a linear matroid and $T \subseteq \binom{V}{s}$. We can find, in polynomial time, a representative set $T^* \subseteq T$ for T in M such that $|T^*| = \mathcal{O}(r(M)^s)$.

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Application: Vertex Cover kernel

Recall the most classical VERTEX COVER kernel.
VERTEX COVER: Given $G = (V, E)$ and k, does G have a vertex cover of size at most k?

Buss' vertex cover kernel

- $d(v) > k$: must include v
- $d(v) = 0$: discard v
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Vertex Cover matroid kernel

Let $(G = (V, E), k)$ be a Vertex Cover instance.

- Matroid $M = (V, \binom{V}{\leq k+2})$ (representable)
- Tuple set $E \subseteq \binom{V}{2}$
- "Query set" $X \subseteq V$, $|X| \leq k$
- $e = \{u, v\} \in E$: e extends X if and only if X misses e ($u, v \notin X$)
- E extends X if and only if X is not a vertex cover
- $E^* \subseteq E$ representative set: E^* reveals every non-vertex cover

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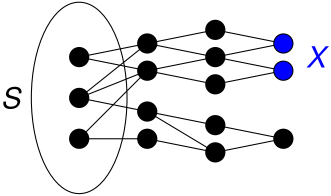
$|E^*| = \mathcal{O}(r(M)^s) = \mathcal{O}(k^2)$.

A gammoid lemma

For a graph $G = (V, E)$, sets $A, B \subseteq V$: Let $\mathcal{C}(A, B)$ be the minimum (A, B)-vertex cut closest to A (may intersect A, B).

Lemma

Let $(G = (V, E), S)$ define a gammoid, $X \subseteq V$ a set linked to S. Then for any $v \in V$, v extends X if and only if S reaches v in $G - \mathcal{C}(S, X)$.



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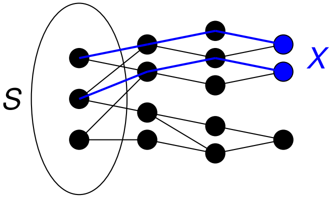

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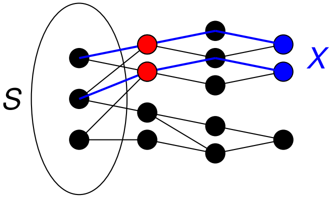

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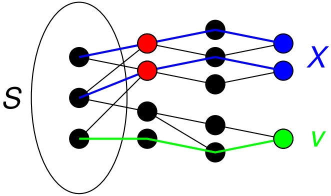

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Proof. (1) v extends X: Path must avoid closest cut.

(2) $X + v$ dependent: $(S, X + v)$-cut, size $|X|$, cuts v from S. This is a min-cut, thus the closest cut also cuts v from S.

The Digraph Pair Cut problem

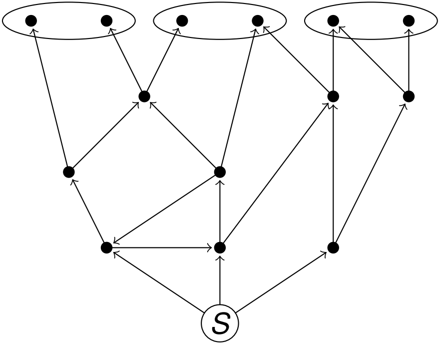
DIGRAPH PAIR CUT

Input: Digraph $G = (V, E)$, source $s \in V$, integer k, set $P \subseteq \binom{V}{2}$.
Parameter: k
Question: Find k vertices X such that no pair in P is reachable from s in $G - X$.

Pair $\{u, v\}$ reachable: u, v each reachable from s.

Will show: can reduce to $\mathcal{O}(k^2)$ pairs.

Digraph Pair Cut example



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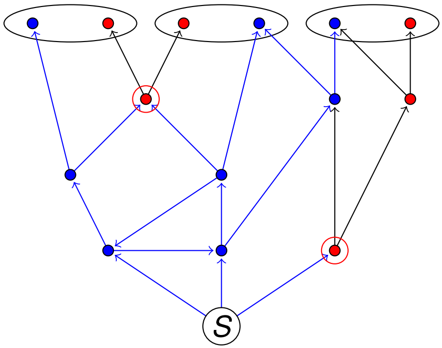

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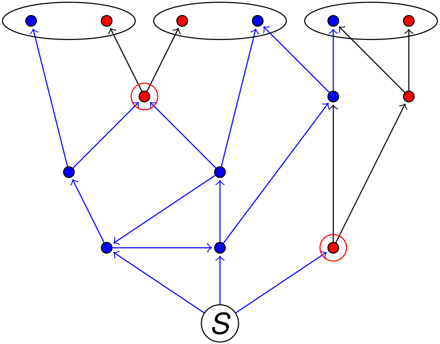

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1. Let $S = k + 1$ copies of s; $T = \emptyset$.
2. Recursively try:
   2.1 Let $X = \mathcal{C}(S, T)$ (first round $X = \emptyset$).
   2.2 If $|X| > k$, reject.
   2.3 If X is a solution, return it.
   2.4 Find pair $\{u, v\} \in P$ reachable in $G - X$, recurse with $T = T + u$ and $T = T + v$.

Claim: Correctly solves DIGRAPH PAIR CUT in $\mathcal{O}^*(2^k)$ steps.

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Testing a proposed solution

Consider a set $X \subset V$, $|X| \leq k$.

- $C := \mathcal{C}(S, X)$ is a solution: we are happy.
- Otherwise, pair $\{u, v\} \in P$ reachable in $G - C$.
- Need to test: S reaches u and v in $G - C$.

Recall: Lemma

Given (G, S): v extends X iff S reaches v in $G - \mathcal{C}(S, X)$.

To test: (u extends X) and (v extends X).

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1. Let $M = (G_1, S_1) + (G_2, S_2)$ (disjoint identical copies of (G, S))
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2. Let $T = \{\{u_1, v_2\} : \{u, v\} \in P\}$.
3. Let $X' = X_1 + X_2$ (copies of X in G_1 and G_2)
4. $\{u_1, v_2\} \in T$ extends X' iff u and v each extends X.

Representative set $T^* \subseteq T$ sufficient testing set, $|T^*| = \mathcal{O}(k^2)$.

Testing set ⇒ polynomial kernel

Input: Instance (G, s, k, P) of DIGRAPH PAIR CUT.
Have: Representative pairs $P^* \subseteq P$.

Pairs in P^* are enough to dry-run the algorithm.

1. Initially $T = \emptyset$
2. Find pair $\{u, v\} \in P$ reachable in $G - \mathcal{C}(S, T)$ (matroid test)
3. Recurse $T = T + u$ and $T = T + v$
4. Solution size is now $|T|$

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Digraph Pair Cut kernelization

Input: Instance (G, s, k, P) of DIGRAPH PAIR CUT.

1. Find representative pairs $P^* \subseteq P$ (as seen).
2. Let $T = \bigcup P^*$ be all vertices in P^* ($|T| = \mathcal{O}(k^2)$).
3. Encode sizes of (S, X)-vertex cuts, $X \subseteq T$, in small space.

Matrix encoding: $\mathcal{O}(|S| \cdot |T| \cdot (|S| \log |T|)) = \tilde{\mathcal{O}}(k^4)$ space.

Later: graph with $\mathcal{O}(|S| \cdot |T| \cdot |S|) = \mathcal{O}(k^4)$ vertices.

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Digraph Pair Cut, wrap-up

- Constraints reduction: Find $\mathcal{O}(k^2)$ sufficient constraints (P^*)
- Encoding terminal cuts from S to $\bigcup P^*$ to encode the instance
- Cut-covering sets (next) for a “proper” kernel

Not shown:

- COMPRESSION ALMOST 2-SAT(k) $\Rightarrow$ DIGRAPH PAIR CUT($2k$)
- Gives ALMOST 2-SAT polynomial kernel

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- Compression Almost 2-SAT$(k) \Rightarrow$ Digraph Pair Cut$(2k)$
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Cut-covering sets

Our goal: Find a cut-covering set

Covering (A, B)-cuts

Let $G = (V, E)$, $S, T \subseteq V$, $S \cap T = \emptyset$. We can find a set $X \subseteq V$ of polynomial size such that for any $A \subseteq S$, $B \subseteq T$, X contains a minimum (A, B)-cut.

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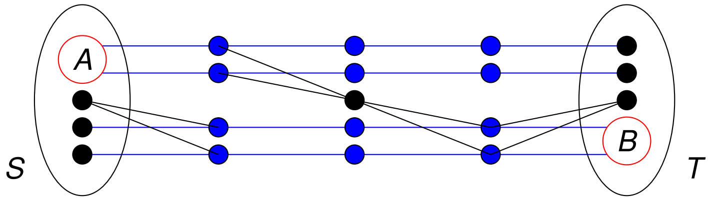

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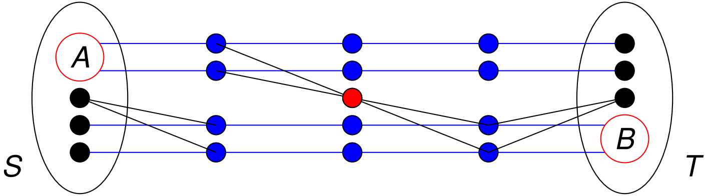

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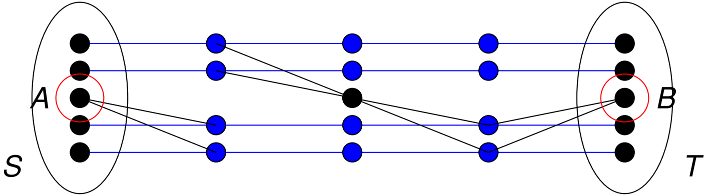

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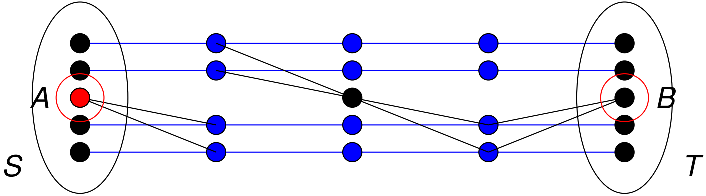

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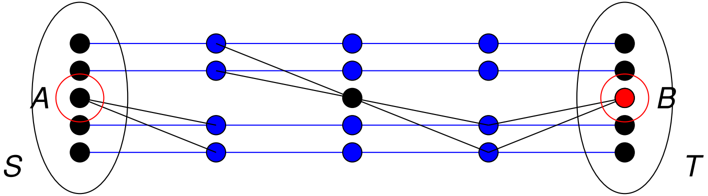

Irrelevant vertices

Let $G = (V, E)$, S, and T be given.

- Let $A \subseteq S$, $B \subseteq T$. A vertex $v \in V$ is essential for (A, B) if v is contained in every minimum (A, B)-cut.
- A vertex $v \in V$ is irrelevant if it is not essential for any (A, B) and not contained in S or T.

If we can find an irrelevant vertex, we can reduce the problem.

Plan:

1. Characterize (A, B)-essential vertices
2. Find them with representative sets query
3. Construct query to find all essential vertices

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Characterizing essential vertices

For any vertex v, let v' be a sink copy: a copy of v with edges oriented inwards.

- $v \in X$, $X + v'$ linked to $S \Leftrightarrow$ vertex-disjoint paths from S to X, with two paths to v

We will show the following.

Let $G = (V, E)$, $A, B \subseteq V$ be given. Let C be a minimum (A, B)-cut. A vertex $v \in V$ is essential for (A, B) if and only if $C + v'$ is linked to A and $C + v'$ is linked to B.

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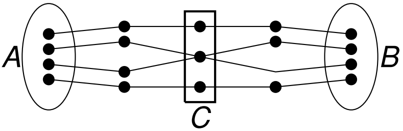

Proof

$G = (V, E)$, $A, B \subseteq V$, minimum cut C as before. To prove:

Vertex v essential for (A, B) iff $C + v'$ is linked to A and to B.

1. A vertex $v \in V$ is (A, B)-essential if and only if $v \in \mathcal{C}(A, C) \cap \mathcal{C}(B, C)$.
   - Almost by definition
2. ...if and only if $C + v'$ is linked to A and to B
   - $C + v'$ linked to A iff v' reachable from A in $G - \mathcal{C}(A, C)$
   - Either $v \in \mathcal{C}(A, C)$, or $\mathcal{C}(A, C)$ cuts v from B
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Representative sets query

We have:

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Tweak it:

Vertex v essential for (A, B) iff v' extends C in (G, A) and in (G, B)

Previous trick:

Vertex v essential for (A, B) iff $v' + v'$ extends $C + C$ in $(G, A) + (G, B)$

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Vertex v essential for (A, B) iff $v' + v'$ extends $C + C$ in $(G, A) + (G, B)$ iff $v' + v'$ extends $(C \cup (S \setminus A)) + (C \cup (T \setminus B))$ in $(G, S) + (G, T)$

1. $M = (G', S) + (G', T)$, with sink copies added to G
2. $P = \{(v'_S, v'_T) : v \in V\}$ (sink copies of v in (G, S) and (G, T))
3. Representative set P^* contains one essential vertex for each pair (A, B)
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Finishing the reduction

Have: $\mathcal{O}(|S| \cdot |T| \cdot \text{cut}(S, T))$ potentially essential vertices.

Let v be an irrelevant vertex. We may make v undeletable:

- Complete $N(v)$ into a clique.
- Delete v from the graph.

This does not change the size of any minimum (A, B)-cut.

Let G' be the result of iterating the above until no vertex is known to be irrelevant. Then $X = V(G')$ is our cut-covering set for the original graph G.

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Consequences, covering min-cuts

Covering (A, B)-cuts

Let $G = (V, E)$ be a possibly directed graph, $S, T \subseteq V$ with $S \cap T = \emptyset$. We can find a set $X \subseteq V$ of size $\mathcal{O}(|S| \cdot |T| \cdot \text{cut}(S, T))$ such that for any $A \subseteq S$,$B \subseteq T$, X contains a minimum (A, B)-cut.

Covering multiway cuts

Let $G = (V, E)$ be an undirected graph, $T \subseteq V$ a set of terminals, and s a constant. We can find a set $X \subseteq V$ of size $\mathcal{O}(|T|^{s+1})$ such that for every partition P of T into at most s sets, X contains a multiway cut of P.

Consequences, kernels

The following problems have polynomial kernels using these methods:

- Graph bipartization problems (edge/vertex deletion)
- ALMOST 2-SAT / VERTEX COVER ABOVE MATCHING
- MULTIWAY CUT WITH DELETABLE TERMINALS
- MULTIWAY CUT, MULTICUT with $\mathcal{O}(1)$ terminals
- GROUP FEEDBACK VERTEX SET, fixed group

Conclusion

Matroid theory and the representative sets lemma have powerful uses in kernelization (we saw only two).

Open questions:

1. Deterministic results (currently: failure risk $\mathcal{O}(2^{-n})$)
2. Non-magic (graph specific) proofs of the results
3. Open kernelization questions:
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