The Ellipsoid method

$\min\{c^T x \mid Ax \geq b\}$

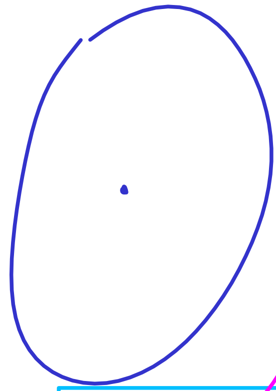
Is center feasible? Yes: done.

No: find violated ineq.

$$a_i^T z < b_i$$

$\forall x \in S : a_i^T x \geq b_i > a_i^T z$

$E :=$ smallest ellipsoid containing



halfspace

$E \cap \{x \mid a_i^T x \geq a_i^T z\}$

Assumption: $\mathrm{vol}(S) \geq 2^{-(n+1)L}$ or $S = \emptyset$

If $P \neq \emptyset$, then $\mathrm{vol}(S) \geq 2^{-(n+1)L}$

$S :=$ set of feasible points $(Ax \geq b)$ inside ball of radius 4^{nL} centered at 0

= initial ellipsoid

Ellipsoid method shrinks ellipsoid in every step but keeps S inside!

Questions: ✓ ① What is the smallest ellipsoid containing ?

✓ ② What is its volume compared to E?

✓ ③ How many steps do we need?

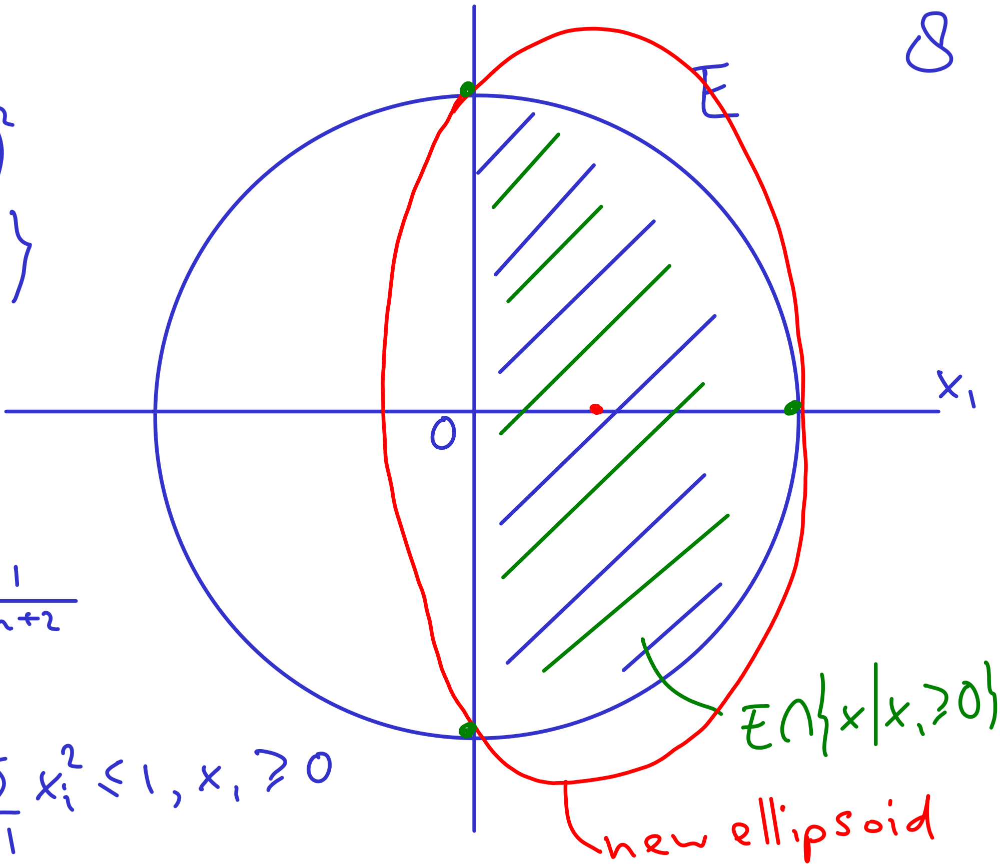
④ Why is assumption reasonable?

⑤ How do we solve LPs?

We will focus on a special case: $E =$ unit ball centered at 0 and halfspace is $\{x \mid x_1 \geq 0\}$

Formula for new ellipsoid:

$$E' = \left\{ x \,\middle|\, \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \le 1 \right\}$$



Answers to Q1, Q2:

Lemma

$E \cap \{x \mid x_1 \ge 0\} \subseteq E'$

and $\dfrac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \le e^{-\frac{1}{2n+2}}$

Proof

Consider x with $\sum_{i=1}^{n} x_i^2 \le 1,\ x_1 \ge 0$

$x \in E'$?

$$\left(\frac{n+1}{n}\right)^2 \left(x_1^2 - \frac{2}{n+1} x_1 + \frac{1}{(n+1)^2}\right) + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \overset{?}{\le} 1$$

$$= \frac{1}{n^2} \left((n^2+2n+1) x_1^2 - (2n+2) x_1 + 1 + (n^2-1) \sum_{2}^{n} x_i^2 \right)$$

$$= \frac{1}{n^2} \left(\underbrace{(2n+2)(x_1^2 - x_1)}_{\le 0} + 1 + (n^2-1) \underbrace{\sum_{i=1}^{n} x_i^2}_{\le 1} \right)$$

≤ 1.

Volume of an ellipsoid is proportional to lengths of axes

E: all axes have length 1

E': one axis has length $\frac{n}{n+1}$

$n-1$ axes have length $\sqrt{\frac{n^2}{n^2-1}}$

$$E' = \left\{ x \,\middle|\, \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \le 1 \right\}$$

$$\frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} = \frac{n}{n+1}\cdot\left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}}$$ take logs

$$= \exp\left(\ln\left(1-\frac{1}{n+1}\right)+\frac{n-1}{2}\cdot\ln\left(1+\frac{1}{n^2-1}\right)\right)$$

$\ln(1+s) \leq s \ \forall\ s > -1$

$$\leq \exp\left(-\frac{1}{n+1}+\frac{n-1}{2}\cdot\frac{1}{n^2-1}\right)$$

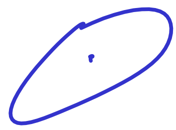
$$= \exp\left(-\frac{1}{n+1}+\frac{1}{2(n+1)}\right) = e^{-\frac{1}{2n+2}} \qquad \blacksquare$$

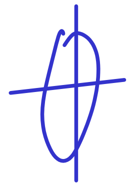
E' is significantly smaller than E!

We do not show that E' = smallest ellipsoid that we could use

What do we do with an arbitrary ellipsoid E_0?

1. Translate space so that z (center of E_0) is moved to the origin
2. Rotate space to align axes of E with coordinate axes
3. Scale coordinates such that E = unit ball
4. Rotate space such that halfspace $= \{x \mid x_1 \geq 0\}$
5. Now we know E'
6. Apply 4–1 to E' to derive E_0'





Step 3 changes volume but NOT ratio of volumes!

$$\frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} = \frac{\mathrm{vol}(E_0')}{\mathrm{vol}(E_0)}$$

Q3. Original ellipsoid: radius 4^{n^2L}, vol $< 8^{n^2L}$ 8

Factor $e^{-\frac{1}{2n+2}}$ decrease per step

Stop if volume $< 2^{-(n+1)L}$ using our assumption

Continue as long as $e^{-\frac{k}{2n+2}}\cdot 8^{n^2L} \geq 2^{-(n+1)L}$
(k steps)

take logs $k\cdot\log_2\left(\exp\left(-\frac{1}{2n+2}\right)\right) + 3n^2L \geq -(n+1)L$

$$k \leq \frac{-(n+1)L - 3n^2L}{\log_2\left(\exp\left(-\frac{1}{2n+2}\right)\right)} = O(n^3L)$$

Note: calculations require taking roots!
$\Longrightarrow$ loss of precision

this bound is incorrect!

Possible to show: sufficient to use 8^L bits of precision

U = abs. value of largest entry of A and b

$L = n(1 + \log n + \log U)$