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Exercises for Optimization

http://www.mpi-inf.mpg.de/departments/d1/teaching/ss12/OPT

Exercise sheet 10 Due: **Tuesday, July 10, 2012**

You need to collect at least 50% of all points on the first six exercise sheets, and at least 50% of all points on the remaining exercise sheets. You are allowed to hand in homework in teams of two.

Exercise 1 (*10 points*)

Consider a directed graph in which each arc is associated with a cost c_{ij}. For any directed cycle, we define its mean cost as the sum of the costs of its arcs, divided by the number of arcs. To guarantee termination of the negative cost cycle algorithm, we are interested in a directed cycle whose mean cost is minimal. We assume that there exists at least one directed cycle.

Consider the linear programming problem $\max\{\lambda \mid p_i + \lambda \le p_j + c_{ij} \text{ for all arcs } (i,j)\}$.

a) Show that this maximization problem is feasible.
Hint: use a minimum spanning tree to determine values for p_i. Use a value for λ to "correct" inequalities whenever there is a cycle. What value of λ is sufficient?

b) Show that if (λ, p) is a feasible solution to the maximization problem, then the mean cost of every directed cycle is at least λ.
Hint: consider an arbitrary cycle. Use the inequalities of the LP relating to this cycle in sequence.

c) Show that the maximization problem has an optimal solution.

d) (extra credit, 10 points) Show how an optimal solution to the maximization problem can be used to construct a directed cycle with minimal mean cost.

Exercise 2 (*10 points*)

Recall that an ellipsoid E can be defined as

$$E(z, D) = \{x \in \mathbb{R}^n | (x - z)^T D^{-1} (x - z) \leq 1\}, \tag{1}$$

where $z \in \mathbb{R}^n$ and D is an $n \times n$ positive definite symmetric matrix.

In class we considered the ellipsoid $S = \{x \in \mathbb{R}^n | \, ||Z^{-1}(x-z)|| \leq \beta\}$ where $Z = \text{diag}(z_1, \ldots, z_n)$, $z > 0$ and $Az = b$. Show that this is indeed an ellipsoid by writing it in the form of (1).

Exercise 3 (*10 points (BT 7.2)*)

Consider a wood company that owns M forest units and wants to find an optimal cutting schedule over a period of K years. Forest unit i is predicted to have $a_i j$ tons of wood available for harvesting during year j. The company wants to meet a demand of d_j tons during year j. However, due to capacity limitations, it can only harvest up to u_j tons during that year. Wood harvested in past years can be sotred and used to meet demand in subsequente years, but there is a cost of c_j for storing one ton of wood between year $j-1$ and j. We also assume that wood that is available but not harvested during a year remains available for harvesting in later years. Formulate the problem of determining a minimum cost harvesting schedule that meets the demand as a network flow problem.

Exercise 4 (*10 points (BT 8.8)*)

Given an undirected graph $G = (N, E)$, two special nodes $s, t \in N$ and weights c_e for all edges $e \in E$, we would like to find a minimum weight set of edges that intersects every path from s to t. Let $\mathbb{K}$ be the set of all such paths. One way of formulating this problem is as follows: $\min\{\sum_{e \in E} c_e x_e | \sum_{e \in K} x_e \geq 1 \forall K \in \mathbb{K}, 0 \leq x_e \leq 1 \forall e \in E\}$. It turns out that the extreme points of the feasible set are vectos with 0-1 coordinates, and that an optimal basic feasible solution corresponds to a minimum cut. Write the associated separation problem as a shortest path problem.